Spectral analysis of Schrödinger operators in non-separable Hilbert spaces

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Estratto

FUNCTIONAL INTEGRATION WITH EMPHASIS ON THE FEYNMAN INTEGRAL
PROCEEDINGS OF A WORKSHOP HELD AT THE UNIVERSITY OF SHERBROOKE SHERBROOKE, QUEBEC CANADA, JULY 21-31, 1986

Supplemento ai Rendiconti del Circolo Matematico di Palermo

Serie II - numero 17 - 1987

Via Archirafi, 34 - 90123 Palermo (Italia)
SPECTRAL ANALYSIS OF SCHröDINGER OPERATORS
IN NON-SEPARABLE HILBERT SPACES

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1. Introduction

Any physical system with $n$ degrees of freedom can be described in terms of $n$ pairs $q_i$, $p_i$ of canonical variables, where $q_1$, ..., $q_n$ are the so-called position variables and $p_1$, ..., $p_n$ the so-called momentum variables of the system. From the classical point of view the state of the system is determined by the results of measuring the $q_i$ and $p_i$ at a given instant. However, as subtle experiments show, the measurements of the $q_i$ and $p_i$ performed repeatedly in a short period of time provide statistical data such that each product $\sigma_{q_i} \sigma_{p_i}$ of the corresponding dispersions $\sigma_{q_i}$, $\sigma_{p_i}$ is not less than a small but non-zero universal constant. The impossibility of effective experimental determination of the simultaneous positions and momenta of the system necessitates a suitable modification of the notion of state. It turns out that a physically reasonable modification is obtained by identifying the state of the system with an ample collection of the experimental mean values of functions of the canonical variables. Quantum mechanics sets up a correspondence between this phenomenological notion of state and some mathematical concepts that appears within the framework of the so-called formalism of non-commuting canonical variables. We shall discuss this identification in more detail.

There is a unique, up to *-isomorphism, C*-algebra $A_n^\tau$ with unit $e$ which is generated, as a C*-algebra, by two subsets $\{u(t) : t \in \mathbb{R}^n\}, \{v(t) : t \in \mathbb{R}^n\}$ such that

1. $u(0) = v(0) = e$,

2. $u(s)u(t) = u(s+t)$ for $s, t \in \mathbb{R}^n$,

3. $v(s)v(t) = v(s+t)$ for $s, t \in \mathbb{R}^n$.

This paper is in final form and no version of it will be submitted for publication elsewhere.
(4) \( u^*(t) = u(-t) \) for \( t \in \mathbb{R}^n \),

(5) \( v^*(t) = v(-t) \) for \( t \in \mathbb{R}^n \),

(6) \( u(s)v(t) = e^{is \cdot t}v(t)u(s) \) for \( s, t \in \mathbb{R}^n \), where \( s \cdot t = s_1t_1 + \cdots + s_n t_n \)

(cf.[1], Th.5.2.8). \( A_n \) is called the canonical commutation relations algebra over \( \mathbb{R}^n \). A state \( \omega \) over \( A_n \) is a linear bounded functional such that 
\[ ||\omega|| = 1 = \omega(e) \]. Let \( E_{A_n} \) be the set of all states over \( A_n \). \( E_{A_n} \) is convex and, under the *-weak topology, it is compact. By the Krein-Milman theorem, the set \( E(E_{A_n}) \) of all extreme points of \( E_{A_n} \) is not empty. Any element of \( E(E_{A_n}) \) is called a pure state over \( A_n \). Via the GNS construction (cf.[1], Th. 2.3.16), with each \( \omega \in E_{A_n} \) one can associate an essentially unique triple \( (H_\omega, \Omega_\omega, \pi_\omega) \), where \( H_\omega \) is a Hilbert space, \( \Omega_\omega \) is an element of \( H_\omega \), and \( \pi_\omega \) is a *-representation of \( A_n \) in \( H_\omega \) such that

(1) the space \( \{ \pi_\omega(A)\Omega_\omega : A \in A_n \} \) is dense in \( H_\omega \) (cyclicity of \( \Omega_\omega \) and \( \pi_\omega \)),

(2) \( \omega(B^*A) = (\pi_\omega(A), \pi_\omega(B)\Omega_\omega) \) for \( A, B \in A_n \).

If \( \omega \) is a pure state over \( A_n \), then the corresponding cyclic representation \( \pi_\omega \) is irreducible. A state \( \omega \) over \( A_n \) is called regular if the representations \( t \mapsto \pi_\omega(u(t)) \) and \( t \mapsto \pi_\omega(v(t)) \) of \( \mathbb{R}^n \) in \( H_\omega \) are strongly continuous. We let \( r(E_{A_n}) \) denote the set of all regular states over \( A_n \).

Quantum mechanics is based on the postulate that each phenomenological state of a system with \( n \) degrees of freedom can be identified with an element \( \omega \) of \( E(E_{A_n}) \cap r(E_{A_n}) \) in such a way that, in suitable units, for all \( s, t \in \mathbb{R}^n \), the experimental mean value of \( e^{is \cdot p + it \cdot q} \)

\[ = \frac{1}{2} \omega(u(s)v(t) + v(t)u(s)) \],

where \( q = (q_1, \ldots, q_n) \) and \( p = (p_1, \ldots, p_n) \). The assumption that \( \omega \) is not a mixture of distinct states over \( A_n \) reflects the fact that the phenomenological state is supposed to be determined with maximal accuracy. The regularity assumption is of technical character and is introduced to make it possible to define the quantum mechanical positions \( \xi^{(n)}_i \) and momenta \( \pi^{(n)}_i \) (non-commuting canonical variables) as the products of \( 1/\sqrt{-1} \) and the infinitesimal generators of the groups \( t \mapsto \pi^{(n)}(u_4(t)) \) and \( t \mapsto \pi^{(n)}(u_4(t)) \), respectively, where
That this definition of the $Q_1^\omega$ and $P_1^\omega$ does not depend on $\omega$ is a consequence of the Stone-von Neumann theorem stating that for any $\omega, \omega' \in E(\mathcal{E}_\mathcal{A}_n) \cap r(\mathcal{E}_\mathcal{A}_n)$ one can identify $H_\omega$ with $H_{\omega'}$ and $\pi_\omega$ with $\pi_{\omega'}$ (cf. [I], Cor. 5.2.15). The essentially unique cyclic representation $\pi$ of $\mathcal{A}_n$, associated with the pure regular states over $\mathcal{A}_n$, is called the regular irreducible representation of $\mathcal{A}_n$. It can be realized in $L^2(\mathbb{R}^n)$ as the unique representation of $\mathcal{A}_n$ satisfying

$$\pi(u(t))f = T_tf,$$

$$\pi(v(t))f = e_t f,$$

(t $\in \mathbb{R}^n$, f $\in L^2(\mathbb{R}^n)$)

where $T_tf(x) = f(x+t)$ and $e_t(x) = e^{it \cdot x}$ for all $x \in \mathbb{R}^n$. The necessity of introducing the $Q_1^\omega$ and $P_1^\omega$ arises naturally when one wants to describe time evolution of the system. The evolution is governed by a strongly continuous one parameter group $(G_t)_{t \in \mathbb{R}}$ of unitary operators on $L^2(\mathbb{R}^n)$ in such a way that if $\Omega \in L^2(\mathbb{R}^n)$ corresponds to an initial state of the system, then $G_t \Omega$ corresponds to the state after time $t$. The infinitesimal generator of $(G_t)_{t \in \mathbb{R}}$, times $1/\sqrt{-1}$, called the Schrödinger operator or Hamiltonian of the system, completely determines $(G_t)_{t \in \mathbb{R}}$. It takes the form

$$\mathcal{H}_\pi = -(P_1^\omega)^2 + \ldots + (P_n^\omega)^2 + V_{\pi},$$

where $V_{\pi} = \mathcal{V}(Q_1^\omega, \ldots, Q_n^\omega)$ and $V$, called a potential, is a real-valued Borel function on $\mathbb{R}^n$ containing information about all the peculiarities of the system.

If $V$ is a special potential of the form

$$V = \sum_{\lambda \in \mathbb{R}^n} a_\lambda e_\lambda,$$
where \( \overline{a}_\lambda = a_{-\lambda} \in C \) for each \( \lambda \) and the sum extends only over finite number of non-zero \( \lambda \), then the corresponding operator \( V_\pi \) can directly be defined in terms of \( \pi \) as being

\[
\sum_{\lambda \in R^n} a_\lambda \pi(v_1(\lambda_1) \cdots v_n(\lambda_n)).
\]

The latter sum makes sense if \( \pi \) is replaced by any irreducible representation \( \rho \) of \( A_n \). In particular, if \( \rho \) is such that the unitary groups \( t \mapsto \rho(u_1(t)) \) are strongly continuous and, correspondingly, have infinitesimal generators \( p_\rho(i) \), then one can define a generalized Hamiltonian

\[
H_\rho = -[(p_1^{(\rho)})^2 + \cdots + (p_n^{(\rho)})^2] + V_\rho,
\]

where

\[
V_\rho = \sum_{\lambda \in R^n} a_\lambda \rho(v_1(\lambda_1) \cdots v_n(\lambda_n)).
\]

More generally, if there is a net \( ((a_\lambda^{(a)})_{\lambda \in R^n})_{a \in A} \) in \( C^{R^n} \) such that \( a_\lambda^{(a)} = a_{-\lambda}^{(a)} \) for each \( a \) and each \( \lambda \), \( a_\lambda^{(a)} = 0 \) only for finitely many \( \lambda \) when \( a \) is fixed, and the limits

\[
\lim_{\alpha} \sum_{\lambda \in R^n} a_\lambda^{(a)} \pi(v_1(\lambda_1) \cdots v_n(\lambda_n)),
\]

\[
\lim_{\alpha} \sum_{\lambda \in R^n} a_\lambda^{(a)} \rho(v_1(\lambda_1) \cdots v_n(\lambda_n))
\]

exist in the sense of strong convergence, then the operators \( V_\pi \) and \( V_\rho \) defined by these limits give rise to two Hamiltonians \( H_\pi \) and \( H_\rho \). While \( H_\rho \) seems to have no direct physical interpretation, it is conceivable that there may exist a mathematically interesting relationship between \( H_\pi \) and \( H_\rho \). In the sequel, we shall provide a few examples that will substantiate this supposition. First, however, we shall take a closer look at irreducible representations of \( A_n \).

2. Irreducible representations of \( A_n \).

Among representations of \( A_n \) the most interesting from our point of view are semiregular ones. These are such non-regular representations of \( \rho \) of \( A_n \) for which each group \( t \mapsto \rho(u_1(t)) \) is strongly continuous. Any representation of \( A_n \) that is neither regular nor semiregular will be called irregular.

The basic example of a semiregular irreducible representation of \( A_n \) involves the Bohr compactification \( bR^n \) of \( R^n \). \( bR^n \) is a compact Abelian group with the property that there is a continuous injective homomorphism \( \alpha: R^n \to bR^n \).
such that \( \alpha(\mathbb{R}^n) \) is dense in \( \mathbb{R}^n \) and, for each \( t \in \mathbb{R}^n \), there is a continuous character \( \chi_t \) of \( \mathbb{R}^n \) such \( \chi_t \circ \alpha = e_t \). Let \( \mathcal{P} \) be the normalized Haar measure on \( \mathbb{R}^n \) and \( L^2(\mathbb{R}^n) \) be the corresponding Hilbert space based on \( \mathcal{P} \). One verifies easily that the representation \( \sigma \) of \( A_n \) in \( L^2(\mathbb{R}^n) \) uniquely determined by

\[
\sigma(u(t))F = T_{\alpha(t)}F,
\]

\[
\sigma(v(t))F = \chi_t F
\]

is semiregular and irreducible.

The Hilbert space in which \( \sigma \) acts is not separable (\( \{\chi_t : t \in \mathbb{R}^n\} \) is a complete orthonormal set in \( L^2(\mathbb{R}^n) \)). There exist semiregular irreducible representations of \( A_n \) acting in separable Hilbert spaces. For example, if \( \gamma \) is a non-measurable character of \( \mathbb{R}^n \), then the representation \( \eta \) of \( A_n \) determined by

\[
\eta(u(t)) = \pi(u(t)),
\]

\[
(\eta \in \mathbb{R}^n)
\]

\[
\eta(v(t)) = \gamma(t) \pi(v(t))
\]

is semiregular, irreducible, and acts in \( L^2(\mathbb{R}^n) \).

However, if a semiregular irreducible representation of \( A_n \) is weakly measurable (as it is in the case of \( \sigma \)), then it must necessarily act in a non-separable Hilbert space (cf. [12]). Weak measurability must be assumed if one wants to use integration theory, thereby the appearance of non-separable Hilbert spaces is inevitable in the context of generalized Schrödinger operators.

In a coming section we shall encounter weakly measurable semiregular irreducible representations of \( A_n \) other then \( \sigma \). We conclude this section by exhibiting an irregular irreducible representation of \( A_n \). An example of such a representation is provided by \( \tau \) acting in \( L^2(\mathbb{R}^n) \) by the rule

\[
\tau(u(t))F = e^{i\frac{|t|^2}{2}} \chi_t \alpha(t) F,
\]

\[
(F \in L^2(\mathbb{R}^n), t \in \mathbb{R}^n; \quad |t|^2 = t_1^2 + \ldots + t_n^2)
\]

\[
\tau(v(t))F = \chi_t F.
\]
3. The Burnat-Shubin-Herczyński theorem

Let $\mathbf{V}$ be a real almost periodic function on $\mathbb{R}^n$. Suppose

$$(a^{(k)}_{\lambda})_{\lambda \in \mathbb{R}^n}$$

is a sequence in $c^{\mathbb{R}^n}$ such that $a^{(k)}_{\lambda} = a^{(k)}_{-\lambda}$ for each $k$ and each $\lambda$, $a^{(k)}_{\lambda} \neq 0$ only for finitely many $\lambda$ when $k$ is fixed, and

$$\lim_{k \to \infty} \| \mathbf{V} - \sum_{\lambda \in \mathbb{R}^n} a^{(k)}_{\lambda} \mathbf{e}_{\lambda} \|_{\infty} = 0.$$ 

Then the limit

$$\lim_{k \to \infty} \sum_{\lambda \in \mathbb{R}^n} a^{(k)}_{\lambda} v_1(\lambda_1) \cdots v_n(\lambda_n)$$

exists and defines an element $\mathbf{V}$ in $A_n$. If $\rho$ is a semiregular irreducible representation of $A_n$, then $\mathbf{v}_\rho = \rho(\mathbf{V})$ and one can define a corresponding Hamiltonian $H_\rho$.

The following striking result is due to Burnat, Shubin, and Herczyński (cf. [2],[6],[7],[13]).

**Theorem** The spectrum of $H_\rho$ is the same as the spectrum of $H_\pi$.

The recent proof of this theorem given by Krupa and Zawisza [10] uses ultrapowers of Hilbert spaces and operators. We shall briefly sketch this proof.

An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ such that

1. $A \in \mathcal{U}$ and $A \subseteq B \subseteq \mathbb{N}$ implies $B \in \mathcal{U}$.
2. $A, B \in \mathcal{U}$ implies $A \cap B \in \mathcal{U}$,
3. $\emptyset \notin \mathcal{U}$,
4. for each $A \subseteq \mathbb{N}$, either $A \in \mathcal{U}$ or $\mathbb{N} \setminus A \in \mathcal{U}$.

An ultrafilter is free if it contains the complements of all finite subsets of $\mathbb{N}$. Given a bounded sequence $(a_n)_{n \in \mathbb{N}}$ of complex numbers and a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, there is $a \in \mathbb{C}$ such that

$$\{ n \in \mathbb{N} : |a - a_n| < \varepsilon \} \in \mathcal{U}$$

for each $\varepsilon > 0$. $a$ is called the limit of $(a_n)_{n \in \mathbb{N}}$ with respect to $\mathcal{U}$ and is denoted by $\lim_{\mathcal{U}} a_n$. If $H$ is a Hilbert space and $L^\infty(H)$ is the Banach
space of bounded sequences in $H$, then
\[
\{(x_n)_{n \in \mathbb{N}} \in l^\infty(H) : \lim_{u} ||x_n|| = 0\}
\]
is a closed subspace of $l^\infty(H)$ and the quotient Banach space
\[
H^N/U = l^\infty(H)/\{(x_n)_{n \in \mathbb{N}} \in l^\infty(H) : \lim_{u} ||x_n|| = 0\}
\]
can be converted into a Hilbert space by taking
\[
[(x_n)_{n \in \mathbb{N}}], [(y_n)_{n \in \mathbb{N}}] = \lim_{u} (x_n, y_n)
\]
as scalar product. $H^N/U$ is called the ultrapower of $H$ with respect to $U$.
For any bounded operator $T$ in $H$, $T^N/U$ defined as
\[
(T^N/U)[(x_n)_{n \in \mathbb{N}}] = [(Tx_n)_{n \in \mathbb{N}}]
\]
is a bounded operator in $H^N/U$; it is called the ultrapower of $T$ with respect to $U$. If $\lambda$ is in the resolvent set of $T$ and $R_\lambda(T)$ is the resolvent of $T$ at $\lambda$, then
\[
R_\lambda(T)(\lambda I - T) = (\lambda I - T)R_\lambda(T) = I \quad (I \text{ the identity operator})
\]
and passing to ultrapowers gives
\[
(R_\lambda(T)^N/U)(\lambda I - T^N/U) = (\lambda I - T^N/U)(R_\lambda(T)N/U) = I,
\]
which shows that $\lambda$ is also in the resolvent set of $T^N/U$ and
$R_\lambda(T^N/U) = R_\lambda(T)^N/U$. Hence, letting $\sigma(A)$ denote the spectrum of the operator $A$, we have $\sigma(T^N/U) \subset \sigma(T)$. If $E$ is a reducing subspace for $T^N/U$, then
$\sigma((T^N/U)|E) = \sigma(TN/U)$ and so $\sigma((T^N/U)|E) \subset \sigma(T)$.

It can be shown (this is the main technical ingredient of the proof) that there exist two isometries $j: L^2(br^n) \to L^2(r^n)^N/U$ and $k: L^2(r^n) \to L^2(br^n)^N/U$ such that, for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $j(L^2(br^n))$ is a reducing subspace for $R_\lambda(H_o)^N/U$, and
\[
jR_\lambda(H_o)j^{-1} = R_\lambda(H_o)^N/U \quad \text{on} \quad j(L^2(br^n))
\]
\[
kR_\lambda(H_o)k^{-1} = R_\lambda(H_o)^N/U \quad \text{on} \quad k(L^2(r^n))
\]
First of these equalities guarantees that $\sigma(R_\lambda(H_o)) \subset \sigma(R_\lambda(H_o))$, the other
implies that \( \sigma(R_\lambda(H_\pi)) = \sigma(R_\lambda(H_\sigma)) \). Thus \( \sigma(R_\lambda(H_\pi)) = \sigma(R_\lambda(H_\sigma)) \) and finally, by the spectral mapping theorem, \( \sigma(H_\pi) = \sigma(H_\sigma) \).

By way of application, let us consider the classical problem of qualitative determination of the spectrum of a one-dimensional Schrödinger operator with a periodic potential with period \( 2\pi \). For each \( \lambda \in [0,1] \), let \( E_\lambda \) be the closed subspace of \( L^2(\mathbb{R}) \) spanned by \( \{ x_{\lambda+n} : n \in \mathbb{Z} \} \). It can be shown without much difficulty that each \( E_\lambda \) is an invariant subspace for \( H_\pi \) and that \( H_\sigma|E_\lambda \) has compact resolvent. Thus each \( \sigma(H_\pi|E_\lambda) \) is a discrete subset of \( \mathbb{R} \) unbounded from above. As \( \lambda \) varies continuously over \( [0,1] \), \( \sigma(H_\pi|E_\lambda) \) also varies continuously, and since the \( \sigma(H_\pi|E_\lambda) \) sum up to yield \( \sigma(H_\pi) \), the latter set has a characteristic band structure. By the Burnat-Shubin-Herczyński theorem, the same is true of \( \sigma(H_\sigma) \).

It is worth noting that while the sets \( \sigma(H_\pi) \) and \( \sigma(H_\sigma) \) coincide, the types of the spectra of \( H_\pi \) and \( H_\sigma \) are different. Indeed, a careful analysis shows that \( H_\pi \) has purely absolutely continuous spectrum, whereas \( H_\sigma \) has pure point spectrum.

As another consequence of the Burnat-Shubin-Herczyński theorem, we mention the spectral mixing theorem (cf. [7]). It states that for any real almost periodic functions \( V_i(1 \leq i \leq K) \) on \( \mathbb{R}^\mathbb{N} \) and any mollified characteristic functions \( \rho_i(1 \leq i \leq K) \) of disjoint cones in \( \mathbb{R}^\mathbb{N} \), if we let \( H_\pi^{(i)} \) be the Schrödinger operators with potentials \( V_i \) and \( H_\pi^{\text{mix}} \) be the Schrödinger operator with potential \( \sum_{i=1}^{K} \rho_i V_i \), then

\[
\bigcup_{i=1}^{K} \sigma(H_\pi^{(i)}) \subset \sigma(H_\pi^{\text{mix}}).
\]

4. First order analogues of \( H_\sigma \)

The spectral analysis of \( H_\sigma \) for a general almost periodic potential is a challenging problem. One can get some idea of the complexity of such analysis by examining first order analogues of \( H_\sigma \).

Let \( V \) be a real almost periodic function on \( \mathbb{R} \). Let \( p(\pi) \) and \( p(\sigma) \) be the momentum operators associated with the representations \( \pi \) and \( \sigma \), respectively, of \( A_1 \). Let

\[
A_\pi = p(\pi) + V_\pi
\]
and

\[ A_\sigma = p^{(\sigma)} + V_\sigma. \]

Let \( U \) be the unitary operator in \( L^2(\mathbb{R}) \) given by

\[ Uf = uf \quad (f \in L^2(\mathbb{R})), \]

where

\[ u(x) = \exp(-i \int_0^x V(t) dt) \quad (x \in \mathbb{R}). \]

Then \( A_\sigma = U_\sigma^* U_\sigma \). In particular, \( A_\sigma \) has purely absolutely continuous spectrum. A similar statement for \( A_0 \) is false. It turns out that the spectrum of \( A_\sigma \) is either pure point, or purely singular continuous, or purely absolutely continuous, and that each of these cases can occur (cf. [5]). Determination of the type of the spectrum of \( A_\sigma \) is related to some cohomology theory which proved to be important in harmonic analysis and ergodic theory (cf. [5]). By using the Trotter product formula, one can show that the unitary group \( (U_t)_{t \in \mathbb{R}} \) generated by \( iA_\sigma \) takes the form

\[ U_t^* F = Y_t \omega(t) \quad (t \in \mathbb{R}, F \in L^2(b\mathbb{R})), \]

where \( Y: (t, \omega) \rightarrow Y_t(\omega) \) is a cocycle on \( b\mathbb{R} \), i.e., a continuous function on \( \mathbb{R} \times b\mathbb{R} \) with values of modulus 1 such that, for all \( s, t \in \mathbb{R} \) and all \( \omega \in b\mathbb{R} \),

\[ Y_{s+t}(\omega) = Y_s(\omega)Y_t(\omega + \alpha(s)) \]

\( Y \) is said to be a coboundary if there exists an invariant section of \( Y \), i.e., a measurable function \( X \) on \( b\mathbb{R} \) with values of modulus 1 such that, given \( t \in \mathbb{R} \),

\[ Y_t(\omega) = X(\omega)X(\omega + \alpha(t)) \]

for \( \mathscr{P} \)-almost all \( \omega \) in \( b\mathbb{R} \). It can be shown that \( A_\sigma \) has pure point spectrum if and only if \( Y \) is a coboundary.

The complicated character of the spectrum of \( A_\sigma \) is reflected in an unexpected manner in the harmonic-analytic properties of the function \( u \) defined above, which is a generalized eigenfunction of \( A_\sigma \). It can be shown, among
other things, that if $A_0$ has purely continuous spectrum, then, for each $\lambda \in \mathbb{R}$

$$ Fu(\lambda) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u(x)e^{-i\lambda x}dx = 0 $$

(cf.[3],[4]). From the standpoint of the classical harmonic analysis this result is quite astonishing, for the usual constructions of functions $f$ in $L^\infty(\mathbb{R})$ such that $Ff(\lambda) = 0$ for each $\lambda \in \mathbb{R}$

$$ \lim_{T \to \infty} \inf \frac{1}{2T} \int_{-T}^{T} |f(x)|^2dx > 0 $$

are very sophisticated (cf.[8],[9], Chap. 6). We recall that a linear continuous functional $m$ on $L^\infty(\mathbb{R})$ is said to be a Banach mean on $L^\infty(\mathbb{R})$ if it is satisfies the following conditions:

(1) $m(1) = 1 = \|m\|$, 

(2) $m(T_x f) = m(f)$ for $f \in L^\infty(\mathbb{R})$ and $x \in \mathbb{R}$.

As is known, the set of all Banach means on $L^\infty(\mathbb{R})$ has at least the cardinality of the hypercontinuum (cf. [11]). Given a Banach mean $m$ on $L^\infty(\mathbb{R})$, a function $f$ in $L^\infty(\mathbb{R})$, we let $F_m f$ denote the Fourier transform of $f$ with respect to $m$, defined as

$$ F_m f(\lambda) = m(f e^{-\lambda}) \quad (\lambda \in \mathbb{R}). $$

One can show that if $Y$ is a coboundary whose no invariant section is continuous, then the set

$$ \{(F_m u(\lambda))_{\lambda \in \mathbb{R}}: m \text{ a Banach mean on } L^\infty(\mathbb{R})\} $$

is $C$-linearly isomorphic to a closed convex set in $C$ different from a singleton (cf.[3],[4]). The existence of a function in $L^\infty(\mathbb{R})$ with the Fourier transforms with respect to Banach means as above is a very strange phenomenon, and one wonders whether this phenomenon can easily be exhibited without making appeal to $A_0$.

We close this section with a remark on weakly measurable semiregular irreducible representations of $A_0^n$. With $A_0^n$ there is associated a weakly measurable semiregular irreducible representation $\kappa$ of $A_1$ in $L^2(h\mathbb{R})$. 
determined by
\[ \kappa(u(t)) = U_t F, \]
\[ (t \in \mathbb{R}, F \in L^2(b\mathbb{R})) \]
\[ \kappa(v(t)) = x_t F \]
(recall that \((U_t)_{t \in \mathbb{R}}\) is the unitary group generated by \(iA_0\)). Operators \(A_0\) with different types of spectrum lead to inequivalent representations of \(A_1\). Now it is clear that some inequivalent weakly measurable semiregular irreducible representations of \(A_n\) can be constructed by taking the tensor products of semiregular irreducible representations of \(A_1\) of the above form and, possibly, of copies of the regular irreducible representation of \(A_1\).

5. Conclusion

It was our intention in this article to present a circle of ideas related to the spectral analysis of non-standard Hamiltonians. Some problems in this area alluded to above have been solved, but many more have not yet been seriously attacked. Among the latter there are some which we find to be of particular interest for further study:

(1) Determine all, up to equivalence, weakly measurable semiregular irreducible representations of \(A_n\).

(2) Does the Burnat-Shubin-Herczyński theorem remain true if \(\sigma\) is replaced by any weakly measurable semiregular irreducible representation of \(A_n\)?

(3) What is the general relation between non-standard Hamiltonians and the generalized eigenfunctions of usual Hamiltonians?

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