Towards the Rigorous Analysis of Evolutionary Algorithms on Random k-Satisfiability Formulas

Andrew M. Sutton

THE UNIVERSITY of ADELAIDE

Average-case complexity

Worst-case complexity can be overly pessimistic.

Instead of measuring worst-case runtime, measure *expected* runtime over a probability distribution on an ensemble of instances (Levin 1986).

Random k-satisfiability: use some random process to generate a Boolean formula over n variables.

What is the expected runtime $T(n)$ of an algorithm over the set of all such formulas?

This talk

SAT in the EC community

Many empirical results . . . but lack of rigorous runtime analysis

So, many opportunities to make progress!

Talk outline

- Analysis of RLS on planted model
  - makes heavy use of a 1992 paper by Koutsoupias and Papadimitriou.
- Uniform model
  - ideas about how low-density formulas might be easy for simple EAs (w.h.p.)

Some definitions.

A Boolean variable:	$x \in \{true, false\}$
A set of n Boolean variables:	$\{x, y, z, w\}$
A set of *literals*:	$\{x, \bar{x}, y, \bar{y}, z, \bar{z}, w, \bar{w}\}$
A *clause*:	$(x \vee y \vee \bar{z})$
A Boolean *formula*:	$(x \vee y \vee \bar{z}) \wedge (\bar{y} \vee \bar{w} \vee z) \wedge \ldots$

A formula F is *satisfiable* iff there exists an assignment (a mapping $\mathcal{A} : \{x, y, z, w\} \rightarrow \{true, false\}$) such that F evaluates to $true$.

Random 3-SAT

Uniform model: $\Psi^{\mathrm{U}}_{n,m}$

Choose m length-3 clauses uniformly at random (without replacement) from the set of nontrivial clauses on n variables.

Planted model: $\Psi^{\mathrm{P}}_{n,p}$

First, choose an assignment $x^\star$ to n variables uniformly at random. Then, every length-3 clause that is satisfied by $x^\star$ is included with probability p.

Random 3-SAT

Uniform model: example

$$(x_1 \vee x_2 \vee \bar{x}_4) \wedge (\bar{x}_2 \vee \bar{x}_3 \vee x_4)$$

Planted model: example

$$x^\star = (1, 0, 0, 1)$$
$$(x_1 \vee x_2 \vee \bar{x}_4) \wedge (x_2 \vee \bar{x}_3 \vee \bar{x}_1)$$

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Proposition (Chernoff bounds).

Let $X_1, X_2, \ldots, X_n$ be independent Poisson trials such that for $1 \leq i \leq n$, $\Pr(X_i = 1) = p_i$, where $0 < p_i < 1$. Let $X = \sum_{i=1}^{n} X_i, \mu = E(X) = \sum_{i=1}^{n} p_i$. Then the following inequalities hold for any $0 < \delta \leq 1$.

$$\Pr(X \geq (1+\delta)\mu) \leq e^{-\mu\delta^2/3}$$

$$\Pr(X \leq (1-\delta)\mu) \leq e^{-\mu\delta^2/2}$$

Randomized local search

Fitness function

Given a formula F on n variables and m clauses,

$$f : \{0,1\}^n \to \{0,1,\ldots,m\}$$

counts the number of clauses satisfied by an assignment.

Algorithm 1: RLS

Choose $x \in \{0,1\}^n$ uniformly at random;
while *stopping criteria not met* **do**
- $y \leftarrow x$;
- Choose $i \in \{1,\ldots,n\}$ uniformly at random;
- $y_i \leftarrow 1 - x_i$;
- **if** $f(y) > f(x)$ **then** $x \leftarrow y$

**Note the strict inequality in the selection step.*

Potential function

Suppose $x^\star$ is the planted solution to a formula $F \sim \Psi^{\mathrm{P}}_{n,p}$.

We define a potential function $\varphi : \{0,1\}^n \to \{0,1,\ldots,n\}$:

$$\varphi(x) = d(x, x^\star).$$

Definition.

We call an assignment x **bad** if, for some $\epsilon > 0$,

$$\varphi(x) > \left(\frac{1}{2} + \epsilon\right) n.$$

An assignment which is not bad is called **good**.

Randomly chosen solutions are likely to be good

Lemma 1

Suppose x is chosen uniformly at random from $\{0,1\}^n$. Then

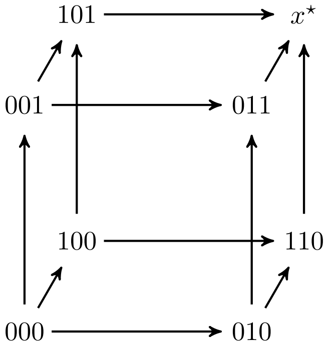
$$\Pr(x \text{ is bad}) \leq e^{-\epsilon^2 n}.$$

Proof.

The probability that $x_i = x_i^\star$ is exactly $1/2$ for all i. Thus let $X_i := [x_i = x_i^\star]$ and the lemma follows from the Chernoff bound with $\mu = n/2$ and $\delta = 2\epsilon$. $\square$

Consider an orientation of a hypercube graph $G = (V, E)$ where $V = \{0, 1\}^n$ and

$$(x, y) \in E \iff \varphi(y) < \varphi(x).$$



The underlying search space

Definition.

We label the directed edge (x, y) in G **deceptive** if $f(x) \geq f(y)$.

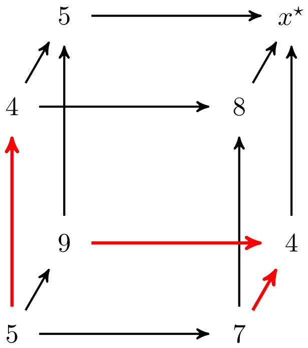
Claim.

1. Each planted solution induces an orientation of G.
2. Each formula F induces a labeling of G.

What is the fraction of edges that are labeled deceptive?

The underlying search space

$G = (V, E)$, with vertices labeled by fitness, deceptive edges in red:



Most "good" edges are nondeceptive

Let $G' = (V', E')$ be the subgraph of G induced by the set of all good nodes:

$$V' = \{x \in \{0,1\}^n : x \text{ is good}\}$$
$$(x', y') \in E' \iff x', y' \in V' \text{ and } \varphi(y) < \varphi(x).$$

Lemma.

Let $(x, y) \in E'$. The probability that (x, y) is labeled deceptive during the construction of F is at most $2e^{-cpn^2}$ for a constant c.

Most “good” edges are nondeceptive

Proof.
Let $S :=$ set of **all clauses** on n variables that are:

- satisfied by $x^\star$
- unsatisfied by x
- satisfied by y

Let $U :=$ set of **all clauses** on n variables that are:

- satisfied by $x^\star$
- satisfied by x
- unsatisfied by y

The edge (x, y) is labeled deceptive iff F has at least as many clauses in U as in S.

Most "good" edges are nondeceptive

Let R_U (R_S) be the number of clauses from U (S) in the formula. Then the probability that (x, y) is labeled deceptive is $\Pr(R_U \geq R_S)$.

R_U (R_S) is a binomial random variable with $|U|$ ($|S|$) trials and probability p.

W.L.O.G., suppose $x^\star = (1, 1, \ldots, 1)$.

Since $d(x, y) = 1$, x and y differ in the (say) i-th bit.

Since x is good, $\varphi(x) \leq (1/2 + \epsilon)n$.

Claim.

$$|S| = \binom{n-1}{2}, \qquad |U| = \binom{n-1}{2} - \binom{n-\varphi(x)}{2}.$$

E.g., $x = 0001, \quad y = 1001, \implies \quad (x_1 \vee x_2 \vee \bar{x}_4) \in S$
E.g., $x = 0101, \quad y = 1101, \implies \quad (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \in U$

Most "good" edges are nondeceptive

So the probability that (x, y) is labeled deceptive is

$$\Pr(R_U \geq R_S) \leq \Pr(R_U \geq a) + \Pr(R_S \leq a) \leq 2e^{-cpn^2}.$$

The inequality comes from the Chernoff bound and c is a constant depending only on ϵ. $\square$

Lemma.

The fraction of deceptive edges in G' is at most $ne^{-n(cpn-\ln 2)}$.

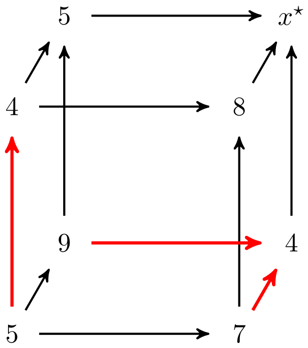
Proof.

By the previous lemma, the fraction of deceptive edges in G' is at most $|E'| \times 2e^{-cpn^2}$ and $|E'| \leq |E| = n2^{n-1}$. $\square$

Claim.

Starting from a random initial solution $x^{(0)}$, RLS gets to a local optimum $\hat{x}$ in expected polynomial time.

We can think of RLS moving through G. RLS moves along an edge (x, y) of G when it replaces the current solution x with a neighboring solution y of better fitness.



Analysis for RLS

Theorem 1.

Probability RLS finds $x^\star$ in expected polynomial time starting from a good initial solution $x^{(0)}$ is at least $1 - ne^{-n(cpn - \ln 2)}$.

Proof.

If $x^{(0)} \in V'$ and RLS never moves along a deceptive edge, then the following are true.

1. RLS never leaves G'
2. $\hat{x} = x^\star$

The probability that an arbitrary edge in G' is deceptive is at most $ne^{-n(cpn - \ln 2)}$. □

Theorem 2.

Suppose $p = \Omega(1/n)$. Then the probability that RLS succeeds on any random planted formula F is $1 - o(1)$.

Proof.

$x^{(0)} \in V'$ w.h.p., and finds $x^\star$ w.h.p. □

The uniform model

Now consider a formula $F \sim \Psi^{\mathrm{U}}_{n,m}$.

Definition.

The **clause density** of F is m/n.

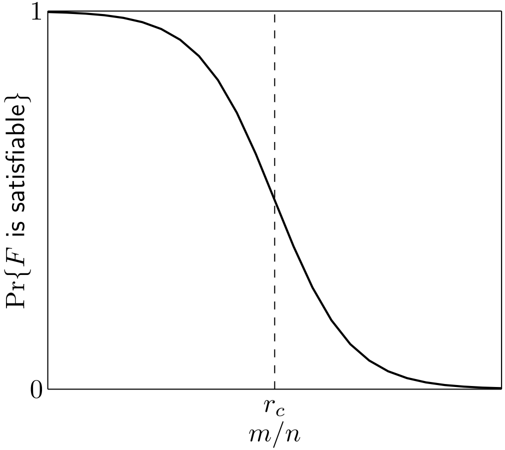
Low-density formulas are *underconstrained*, high-density formulas are *overconstrained*.

Satisfiability threshold conjecture

For all $k \geq 3$ there exists a real number $r_c(k)$ such that

$$\lim_{n\to\infty} \Pr\{F \text{ is satisfiable}\} = \begin{cases} 1 & m/n < r_c(k); \\ 0 & m/n > r_c(k). \end{cases}$$

The uniform model



The pure literal heuristic

Definition.

A literal ℓ is called *pure* in a set of clauses if its negation does not occur in that set.

Example:

$$(x_1 \vee \bar{x}_2 \vee \bar{x}_4) \wedge (\bar{x}_2 \vee x_4 \vee x_3)$$

Pure literals: x_1, $\bar{x}_2$, x_3.

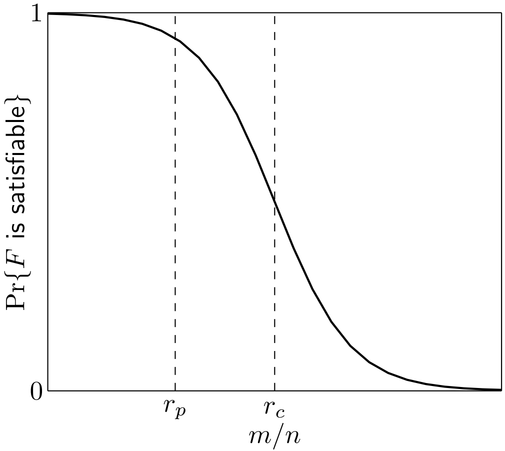
Algorithm 2: The pure literal heuristic (PLH).

while $\mathscr{C}$ *contains pure literals* **do**
    Select a literal ℓ which is pure in $\mathscr{C}$;
    $\ell \leftarrow true$;
    $\mathscr{C} \leftarrow \mathscr{C} \setminus \{C \in \mathscr{C} : \ell \in C\}$;

The pure literal heuristic

Theorem (due to Broder et al., 1993).

Let $F \sim \Psi^{\mathrm{U}}_{n,m}$ be a uniform random 3-SAT formula where $m/n < 1.63$. PLH succeeds on F with probability $1 - o(1)$.



Ideas for the (1+1) EA

If PLH succeeds on a 3-SAT formula F, then PLH must succeed on every subset of clauses from F.

The (1+1) EA can simulate the first step of PLH since, as long as there are pure literals in F, a fitter solution can be obtained by setting them correctly.

Open question.

Can the (1+1) EA efficiently simulate PLH?