Competitive VCG Redistribution Mechanism for Public Project Problem

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Abstract. The VCG mechanism has many nice properties, and can be applied to a wide range of social decision problems. One problem of the VCG mechanism is that even though it is efficient, its social welfare (agents' total utility considering payments) can be low due to high VCG payments. VCG redistribution mechanisms aim to resolve this by redistributing the VCG payments back to the agents. Competitive VCG redistribution mechanisms have been found for various resource allocation settings. However, there has been almost no success outside of the scope of allocation problems. This paper focuses on another fundamental model - the public project problem. In Naroditskiy et al. 2012, it was conjectured that competitive VCG redistribution mechanisms exist for the public project problem, and one competitive mechanism was proposed for the case of three agents (unfortunately, both the mechanism and the techniques behind it do not generalize to cases with more agents). In this paper, we propose a competitive mechanism for general numbers of agents, relying on new techniques.

Keywords: VCG redistribution mechanisms · Dominant strategy implementation · Groves mechanisms · Public good provision

1 Introduction

The VCG mechanism [2,3,15] (referring specifically to the Clarke mechanism) has many nice properties. It is *efficient*, *strategy-proof*, and *weakly budget-balanced* (*non-deficit*). It is a general mechanism that can be applied to many different social decision problems.

One problem of the VCG mechanism is that even though it is efficient[1], its social welfare[2] can be low due to high VCG payments. As a result, the VCG mechanism is not suitable for scenarios where we want to maximize the social welfare. One example scenario is that a group of agents may need to allocate among themselves some shared resources (*e.g.*, airlines unsharing take-off/landing slots).

[1] The VCG mechanism always picks the outcome that maximizes the agents' total valuation.

[2] By social welfare, we mean the agents' total utility: total valuation minus total payment.

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Another example scenario is that a group of agents may need to decide among themselves whether or not to build a public project (*e.g.*, community library) that can be accessed by everyone.

In light of the above drawback of the VCG mechanism, the VCG redistribution mechanisms were proposed [1]. These mechanisms would allocate according to the VCG mechanism, but then on top of the VCG payments, the agents also receive *back* some redistribution payments, therefore increasing the social welfare. An agent's redistribution must not depend on her own type, which is to ensure that the redistribution process does not change the agents' incentives. After incorporating redistribution, the overall mechanism remains efficient and strategy-proof (as the original VCG mechanism is efficient and strategy-proof, plus that the agents' incentives do not change). The problem of VCG redistribution mechanism design is essentially designing how to redistribute the VCG payments back to the agents *as much as possible without redistributing too much.* We cannot redistribute too much because if we redistribute more than the total VCG payment, then the mechanism is no longer weakly budget-balanced. In summary, VCG redistribution mechanisms are non-deficit Groves mechanisms.

Formally, given a social decision problem, let the outcome space be O and the number of agents be n. We use Θ_i to denote agent i's type space. For $o \in O$ and $\theta_i \in \Theta_i$, we use $u(\theta_i, o)$ to denote agent i's valuation for outcome o when her type is θ_i.

The VCG mechanism picks the following optimal outcome:

$$o^* = \arg\max_{o \in O} \sum_i u(\theta_i, o)$$

Agent i's VCG payment equals how much her presence hurts the other agents:

$$\max_{o \in O} \sum_{j \neq i} u(\theta_j, o) - \sum_{j \neq i} u(\theta_j, o^*)$$

A VCG redistribution mechanism is characterized by a list of redistribution functions r_i, where $r_i(\theta_{-i})$ represents agent i's redistribution (positive means receiving money). We notice that agent i's redistribution $r_i(\theta_{-i})$ does not depend on agent i's own type, which ensures strategy-proofness and efficiency. To ensure weakly budget balance, we require that the total redistribution $\sum_i r_i(\theta_{-i})$ is at most the total VCG payment.

Moulin [13] proposed the following performance evaluation criterion for VCG redistribution mechanisms. A mechanism's worst-case efficiency ratio is defined as the worst-case ratio (over all type profiles) between the achieved social welfare and the optimal social welfare. The optimal social welfare is the same as the maximum total valuation, which can be achieved by the (omniscient/omnipotent) *first-best* mechanism.

The achieved social welfare of a VCG redistribution mechanism equals

$$\sum_i u(\theta_i, o^*) - \sum_i (\max_{o \in O} \sum_{j \neq i} u(\theta_j, o) - \sum_{j \neq i} u(\theta_j, o^*)) + \sum_i r_i(\theta_{-i})$$

The three terms in the above expression represent the "achieved total valuation under VCG", "the agents' total VCG payment", and "the agents' total redistribution", respectively.

The worst-case efficiency ratio is then (θ represents the type profile):

$$\min_{\theta} \frac{n\sum_i u(\theta_i, o^*) - \sum_i \max_{o\in O}\sum_{j\neq i} u(\theta_j, o) + \sum_i r_i(\theta_{-i})}{\sum_i u(\theta_i, o^*)} \tag{1}$$

The worst-case efficiency ratio is between 0 and 1. Higher ratios correspond to better worst-case performance in terms of social welfare. The original VCG mechanism (not redistributing anything) typically has a worst-case efficiency ratio of 0, or approaching 0 asymptotically, which we will elaborate more later on. In this paper, we study *competitive VCG redistribution mechanisms.*

Definition 1. *A VCG redistribution mechanism is* **competitive** *if its worst-case efficiency ratio is bounded below by a positive constant.*

That is, a VCG redistribution mechanism is competitive if it guarantees a constant fraction of the optimal social welfare in the worst case.

There has been a lot of success on designing competitive VCG redistribution mechanisms in *resource allocation settings.* Actually, a lot of the proposed mechanisms are not only competitive, but also proven to be optimal (you cannot find other mechanisms with higher worst-case efficiency ratios). For example, for multi-unit auctions with unit demand, Moulin [13] identified a competitive mechanism with the optimal ratio. For the slightly more general setting of multi-unit auctions with nonincreasing marginal values, an almost identical result (under a slightly different objective) was independently proposed by Guo and Conitzer [7]. Gujar and Narahari [4] conjectured that the mechanism proposed in Moulin [13] and Guo and Conitzer [7] can be further generalized to heterogeneous item auctions with unit demand. The conjecture was confirmed by Guo [6]. There has also been work on competitive VCG redistribution mechanisms that are not optimal. Guo [5] proposed competitive VCG redistribution mechanisms for combinatorial auctions with gross substitutes valuations.

Despite the success in resource allocation settings, no competitive VCG redistribution mechanisms were identified outside of the scope of resource allocation. The only exception is Naroditskiy et al. [14], where the authors studied the public project problem. The authors derived an upper bound on the worst-case efficiency ratio for the public project setting. The authors also proposed one mechanism whose worst-case efficiency ratio matches the upper bound when there are exactly three agents. Unfortunately, the proposed mechanism and its underlying techniques do not generalize to more than three agents. The authors also proposed a few heuristic-based redistribution mechanisms that seem to perform well based on numerical simulation (unfortunately, the numerical simulation can only handle up to six agents[3]).

[3] Even for n between 4 and 6, there is no guarantee of worst-case performance, because the worst-case is simulated via sampling, which may not be extensive enough.

Guo et al. [9] also studied redistribution for the public project problem. Most results are not directly related to this paper, because the authors there focused on inefficient partitioning-based mechanisms instead of VCG redistribution mechanisms. However, there is one result that is relevant, which is that the original VCG mechanism has a worst-case efficiency ratio of $1/n$ for the public project problem. That is, the original VCG mechanism is not competitive.

In summary, outside of the scope of resource allocation, there are no known competitive VCG redistribution mechanisms. This paper continues the study of public project problem, and proposes the first competitive VCG redistribution mechanism outside of the scope of resource allocation.

2 Model Description

We study the public project problem, which is a classic problem well studied in both computer science and economics [8–12, 14].

There are n agents who need to decide among themselves whether or not to build a public project that can be accessed by everyone (*e.g.*, a bridge). The cost of the project is C. We assume the cost is already there in the beginning, *e.g.*, the government has bestowed C to the community, and the community needs to decide what to do with it. There are two outcomes: (1) build the public project; (2) not build and divide the money evenly (everyone receives C/n). Without loss of generality, we assume $C = 1$.

We use θ_i to represent agent i's valuation for the public project, so an agent's valuation is θ_i if the decision is to build, and her valuation is $1/n$ if the decision is to not build (divide money instead). Without loss of generality [14], we assume θ_i is in $[0, 1]$.

The VCG mechanism chooses to build if and only if the total valuation of the project exceeds the cost. That is, we build if and only if $\sum_i \theta_i \geq 1$. The agents' total valuation under VCG is then $\max\{\sum_i \theta_i, 1\}$. If the VCG decision is to build, then agent i's VCG payment equals $\max\{\sum_{j\neq i} \theta_j, \frac{n-1}{n}\} - \sum_{j\neq i} \theta_j$. If the VCG decision is not to build, then agent i's VCG payment equals $\max\{\sum_{j\neq i} \theta_j, \frac{n-1}{n}\} - \frac{n-1}{n}$. The agents' total VCG payment equals

$$\sum_i \max\{\sum_{j\neq i} \theta_j, \frac{n-1}{n}\} - (n-1)\max\{\sum_i \theta_i, 1\}$$

Based on Expression 1, if we use the r_i to represent the redistribution functions, then the worst-case efficiency ratio equals:

$$\min_{\theta} \frac{n\max\{\sum_i \theta_i, 1\} - \sum_i \max\{\sum_{j\neq i} \theta_j, \frac{n-1}{n}\} + \sum_i r_i(\theta_{-i})}{\max\{\sum_i \theta_i, 1\}} \quad (2)$$

Our task is to design the r_i so that the above ratio is bounded below by a positive constant.

Based on Expression 2, it is easy to see that the VCG mechanism's worst-case efficiency ratio is at most $1/n$. For example, let us consider the profile where $\theta_1 = 1$ and $\theta_i = 0$ for $i > 1$. Expression 2 simplifies to

$$\frac{n - (n-1) - \frac{n-1}{n}}{1} = \frac{1}{n}$$

3 Intuition and Result

Ideally, we want the total redistribution to be as close as possible to the total VCG payment. There are n agents, so it is a reasonable heuristic to try to make sure that every agent's redistribution is as close as possible to $1/n$ times the total VCG payment. The Cavallo mechanism [1] is somewhat based on this idea. The Cavallo mechanism has found a lot of success in the resource allocation settings. It is competitive in all the resource allocation settings mentioned earlier.[4]

We use $VCG(\theta_i, \theta_{-i})$ to represent the total VCG payment. Under Cavallo's mechanism, $r_i(\theta_{-i})$ is defined as

$$\frac{\min_{\theta_i'} VCG(\theta_i', \theta_{-i})}{n}$$

It should be noted that based on the above definition, an agent's redistribution is independent of her own type, and the total redistribution is never more than the total VCG payment.

Unfortunately, the Cavallo mechanism is not competitive for the public project problem, because it never redistributes anything. No matter what is θ_{-i}, we can always find θ_i' so that $VCG(\theta_i', \theta_{-i})$ equals 0.[5]

We notice that for all the resource allocation settings mentioned earlier, we have[6]

$$\min_{\theta_i'} VCG(\theta_i', \theta_{-i}) = VCG(\theta_{-i})$$

$VCG(\theta_{-i})$ represents the total VCG payment when agent i is removed from the system. Given that redistributing to every agent $VCG(\theta_{-i})/n$ resulted in competitive mechanisms for resource allocation settings, what if we do the same for the public project problem?

In the public project setting, if we remove agent i, then there are $n-1$ agents left, who choose between building the project and receiving $1/(n-1)$ each. If we redistribute every agent $VCG(\theta_{-i})/n$, then we have

$$r_i(\theta_{-i}) = \frac{VCG(\theta_{-i})}{n} = \frac{\sum_{j \neq i} \max\{\sum_{k \neq i,j} \theta_k, \frac{n-2}{n-1}\} - (n-2)\max\{\sum_{k \neq i} \theta_k, 1\}}{n}$$

[4] We do need the minor assumption that the number of agents is large compared to the number of items.

[5] If $\sum_{j \neq i} \theta_j \geq \frac{n-1}{n}$, then pick $\theta_i' = 1$. Otherwise, pick $\theta_i' = 0$.

[6] This is called revenue monotonicity.

We numerically simulated the above redistribution functions and were not satisfied with its performance (if it does not even work well numerically, then there is no point investing time trying to prove that it has good worst-case performance). Fortunately, after trials and errors (based on both manual analysis and numerical simulation), we noticed that if we make two minor technical adjustments, we are able to obtain much better redistribution functions. We replace the denominator n by $n-1$, and we replace $\frac{n-2}{n-1}$ by $\frac{n-1}{n}$. At the end, we have

$$r_i(\theta_{-i}) = \frac{\sum_{j \neq i} \max\{\sum_{k \neq i,j} \theta_k, \frac{n-1}{n}\} - (n-2)\max\{\sum_{k \neq i} \theta_k, 1\}}{n-1} \quad (3)$$

We (partly) used numerical simulation to reach the above starting point. We then **mathematically prove** that we can build a competitive VCG redistribution mechanism based on the above functions.

Let θ be a type profile, we define $Diff(\theta)$ as

$$\sum_i r_i(\theta_{-i}) - VCG(\theta)$$

$Diff(\theta)$ represents the difference between the amount redistributed and the total VCG payment. It turns out that we can bound $Diff(\theta)$ as follows:

Proposition 1.

$$\forall \theta, L(n) \leq Diff(\theta) \leq U(n)$$

$$U(n) = \frac{1}{n-1} + \frac{n-1}{4n} + \frac{4(n+1)^3}{27n(n-1)^2}$$

$$L(n) = \min\{\frac{1}{n-1} - \frac{1}{n} - \frac{(n-1)^2}{4n^2}, \frac{1}{n-1} + \frac{1}{2n} - \frac{1}{2}\} - \frac{n-2}{n(n-1)}$$

Theorem 1. *We define $r'_i(\theta_{-i})$ to be $r_i(\theta_{-i}) - U(n)/n$.*

(r_i is defined according to Eq. 3. $U(n)$ is defined according to Proposition 1.)

If we redistribute according the r'_i, then the corresponding VCG redistribution mechanism is competitive.

When n goes to infinity, the worst-case efficiency ratio approaches 0.102.

4 Proof of Proposition 1

For presentation purposes, we introduce the following notation:

- For all i, $X_i = \sum_{j \neq i} \theta_j$ (the sum of the types other than i's own type).
- $X = \sum_i \theta_i = \frac{\sum_i X_i}{n-1}$ (the sum of all the types).

Using the new notation, we have $Diff(\theta)$ equals

$$(n-1)\max\{X,1\} + \frac{1}{n-1}\sum_i\sum_{j\neq i}\max\{X_i+X_j-X,\frac{n-1}{n}\}$$
$$-\sum_i \max\{X_i,\frac{n-1}{n}\} - \frac{n-2}{n-1}\sum_i \max\{X_i,1\} \quad (4)$$

We use $E(X_1, X_2, \ldots, X_n)$ to denote Expression 4. We use the short form E when there is no ambiguity. The set of all possible values of the X_i must be a subset of:

$$\Lambda = \{(X_1, X_2, \ldots, X_n)|\forall i, 0 \le X_i \le X = \frac{\sum_i X_i}{n-1}\}$$

$Diff(\theta)$ is bounded above by $\max_\Lambda E$. Next, we show how to calculate (an upper bound of) $\max_\Lambda E$.

Proposition 2. *Let* $(X_1, X_2, \ldots, X_n)$ *be an arbitrary element of* Λ. $X = \frac{\sum_i X_i}{n-1}$. *Two coordinates* X_i *and* X_j *are said to be* **from the same band** *if*

- $0 \le X_i, X_j \le \min\{\frac{n-1}{n}, X\}$, *or*
- $\frac{n-1}{n} \le X_i, X_j \le \min\{1, X\}$, *or*
- $1 \le X_i, X_j \le X$.

Let X_i *and* X_j *be two coordinates from the same band. Without loss of generality, we assume* $X_i \le X_j$. *We use* $(X_i, X_j, \ldots)$ *to denote the original element* $(X_1, X_2, \ldots, X_n)$ *from* Λ. *We use* $(X_i - \epsilon, X_j + \epsilon, \ldots)$ *to denote the new element where* X_i *is replaced by* $X_i - \epsilon$ *and* X_j *is replaced by* $X_j + \epsilon$ *(*$\epsilon \ge 0$*).*

If $X_i - \epsilon$ *and* $X_j + \epsilon$ *are still from the same band, then*

- $(X_i - \epsilon, X_j + \epsilon, \ldots)$ *is still an element of* Λ.
- $E(X_i, X_j, \ldots) \le E(X_i - \epsilon, X_j + \epsilon, \ldots)$.

In words, if two coordinates X_i and X_j are from the same band, then by "pushing their values apart within their band", the resulting element is still in Λ, and the resulting new value of E does not decrease.

4.1 Upper Bound of E

Our goal is to calculate $\max_\Lambda E$. We recall that Λ is defined as

$$\Lambda = \{(X_1, X_2, \ldots, X_n)|\forall i, 0 \le X_i \le X = \frac{\sum_i X_i}{n-1}\}$$

We notice that Λ is the union of the following three sets:

$$\Lambda_1 = \{(X_1, X_2, \ldots, X_n)|\forall i, 0 \le X_i \le X = \frac{\sum_i X_i}{n-1}, X \le \frac{n-1}{n}\}$$

$$\Lambda_2 = \{(X_1, X_2, \ldots, X_n) | \forall i, 0 \leq X_i \leq X = \frac{\sum_i X_i}{n-1}, \frac{n-1}{n} \leq X \leq 1\}$$

$$\Lambda_3 = \{(X_1, X_2, \ldots, X_n) | \forall i, 0 \leq X_i \leq X = \frac{\sum_i X_i}{n-1}, 1 \leq X\}$$

We certainly have

$$\max_{\Lambda} E = \max\{\max_{\Lambda_1} E, \max_{\Lambda_2} E, \max_{\Lambda_3} E\}$$

Value of $\max_{\Lambda_1} E$. We first analyze $\max_{\Lambda_1} E$. Let $(X_1^*, X_2^*, \ldots, X_n^*)$ be an element in Λ_1 that maximizes E. Let $X^* = \frac{\sum_i X_i^*}{n-1}$. Since $(X_1^*, X_2^*, \ldots, X_n^*) \in \Lambda_1$, $X^* \leq \frac{n-1}{n}$. By symmetry, it is without loss of generality to assume that $X_1^* \leq X_2^* \leq \ldots \leq X_n^*$.

$E(X_1^*, X_2^*, \ldots, X_n^*)$ simplifies to

$$(n-1) + \frac{1}{n-1} \sum_i \sum_{j \neq i} \max\{X_i^* + X_j^* - X^*, \frac{n-1}{n}\}$$

$$-\sum_i \frac{n-1}{n} - \frac{n-2}{n-1} \sum_i 1$$

Since $X_i^* + X_j^* - X^* \leq X^* \leq \frac{n-1}{n}$, the above further simplifies to

$$(n-1) + \frac{1}{n-1} \sum_i \sum_{j \neq i} \frac{n-1}{n} - (n-1) - \frac{n-2}{n-1} n = \frac{1}{n-1}$$

That is, $\max_{\Lambda_1} E = \frac{1}{n-1}$.

Value of $\max_{\Lambda_2} E$. We now analyze $\max_{\Lambda_2} E$. Let $(X_1^*, X_2^*, \ldots, X_n^*)$ be an element in Λ_2 that maximizes E. Let $X^* = \frac{\sum_i X_i^*}{n-1}$. Since $(X_1^*, X_2^*, \ldots, X_n^*) \in \Lambda_2$, $\leq \frac{n-1}{n} \leq X^* \leq 1$. By symmetry, it is without loss of generality to assume that $X_1^* \leq X_2^* \leq \ldots \leq X_n^*$.

Since $\frac{n-1}{n} \leq X^* \leq 1$, the X_i^* fall into two possible bands. They are $[0, \frac{n-1}{n}]$ and $[\frac{n-1}{n}, X^*]$. (One band may be empty.) By Proposition 2, it is without loss of generality to assume that there exists at most one X_i^* that is in $(0, \frac{n-1}{n})$, and there exists at most one X_j^* that is in $(\frac{n-1}{n}, X^*)$. Hence, it is without loss of generality to assume that $(X_1^*, X_2^*, \ldots, X_n^*)$ has the following form:

$$(0, 0, \ldots, 0, [u], \frac{n-1}{n}, \frac{n-1}{n}, \ldots, \frac{n-1}{n}, [v], X^*, X^*, \ldots, X^*)$$

In the above, $[u]$ represents that there is at most one value u in $(0, \frac{n-1}{n})$, and $[v]$ represents that there is at most one value v in $(\frac{n-1}{n}, X^*)$.

- Case 1: There does not exist one value v that is within $(\frac{n-1}{n}, X^*)$. That is, $(X_1^*, X_2^*, \ldots, X_n^*)$ has the following form:

$$(0, 0, \ldots, 0, [u], \frac{n-1}{n}, \frac{n-1}{n}, \ldots, \frac{n-1}{n}, X^*, X^*, \ldots, X^*)$$

Let p be the number of X^*. $E(X_1^*, X_2^*, \ldots, X_n^*)$ simplifies to

$$(n-1) + \frac{1}{n-1}\sum_i \sum_{j \neq i} \frac{n-1}{n} + \frac{1}{n-1} p(p-1)(X^* - \frac{n-1}{n})$$

$$-(n-p)\frac{n-1}{n} - pX^* - \frac{n-2}{n-1}n$$

We notice that the above expression is linear in X^* and it is nonincreasing in X^*. To maximize it, we let $X^* = \frac{n-1}{n}$. The expression simplifies to

$$(n-1) + \frac{1}{n-1}\sum_i \sum_{j \neq i} \frac{n-1}{n} - n\frac{n-1}{n} - \frac{n-2}{n-1}n = \frac{1}{n-1}$$

– Case 2: There does exist one value v that is in $(\frac{n-1}{n}, X^*)$. That is, $(X_1^*, X_2^*, \ldots, X_n^*)$ has the following form:

$$(0, 0, \ldots, 0, [u], \frac{n-1}{n}, \frac{n-1}{n}, \ldots, \frac{n-1}{n}, v, X^*, X^*, \ldots, X^*)$$

Let p be the number of X^*. $E(X_1^*, X_2^*, \ldots, X_n^*)$ simplifies to

$$(n-1) + \frac{1}{n-1}\sum_i \sum_{j \neq i} \frac{n-1}{n} + \frac{1}{n-1} 2p(v - \frac{n-1}{n}) + \frac{1}{n-1} p(p-1)(X^* - \frac{n-1}{n})$$

$$-(n-p-1)\frac{n-1}{n} - v - pX^* - \frac{n-2}{n-1}n \qquad (5)$$

Expression 5 is linear in v. We know $\frac{n-1}{n} \leq v \leq X^*$. Therefore, by replacing v by either $\frac{n-1}{n}$ or X^*, we obtain an upper bound on Expression 5.
Replace v by $\frac{n-1}{n}$: Expression 5 becomes

$$(n-1) + \frac{1}{n-1}\sum_i \sum_{j \neq i} \frac{n-1}{n} + \frac{1}{n-1} p(p-1)(X^* - \frac{n-1}{n})$$

$$-(n-p)\frac{n-1}{n} - pX^* - \frac{n-2}{n-1}n$$

Just like Case 1, the above is nonincreasing in X^*. When $X^* = \frac{n-1}{n}$ (minimized), it equals $\frac{1}{n-1}$.
Replace v by X^*: Expression 5 becomes

$$(n-1) + \frac{1}{n-1}\sum_i \sum_{j \neq i} \frac{n-1}{n} + \frac{1}{n-1} p(p+1)(X^* - \frac{n-1}{n})$$

$$-(n-p-1)\frac{n-1}{n} - (p+1)X^* - \frac{n-2}{n-1}n$$

Again, the above is nonincreasing in X^*. When $X^* = \frac{n-1}{n}$ (minimized), it equals $\frac{1}{n-1}$.

In conclusion, $\max_{\Lambda_2} E = \frac{1}{n-1}$.

Value of $\max_{\Lambda_3} E$. We now analyze $\max_{\Lambda_3} E$. Let $(X_1^*, X_2^*, \ldots, X_n^*)$ be an element in Λ_3 that maximizes E. Let $X^* = \frac{\sum_i X_i^*}{n-1}$. Since $(X_1^*, X_2^*, \ldots, X_n^*) \in \Lambda_3$, $X^* \geq 1$. By symmetry, it is without loss of generality to assume that $X_1^* \leq X_2^* \leq \ldots \leq X_n^*$.

Since $1 \leq X^*$, the X_i^* fall into three possible bands. They are $[0, \frac{n-1}{n}]$, $[\frac{n-1}{n}, 1]$, and $[1, X^*]$. (Some bands may be empty.) By Proposition 2, it is without loss of generality to assume that there exists at most one value in each of the following three intervals: $(0, \frac{n-1}{n})$, $(\frac{n-1}{n}, 1)$, and $(1, X^*)$. That is, $(X_1^*, X_2^*, \ldots, X_n^*)$ has the following form:

$$(0, 0, \ldots, 0, [u], \frac{n-1}{n}, \frac{n-1}{n}, \ldots, \frac{n-1}{n}, [v], 1, 1, \ldots, 1, [w], X^*, X^*, \ldots, X^*)$$

In the above, $[u]$ represents that there is at most one value u in $(0, \frac{n-1}{n})$, $[v]$ represents that there is at most one value v in $(\frac{n-1}{n}, 1)$, and $[w]$ represents that there is at most one value w in $(1, X^*)$.

Proposition 3. *Let $(X_1, X_2, \ldots, X_n)$ be an arbitrary element of Λ_3.*

We use $(X_1', X_2', \ldots, X_n')$ to denote the following new element

$$\left(\max\{X_1, \frac{n-1}{n}\}, \max\{X_2, \frac{n-1}{n}\}, \ldots, \max\{X_n, \frac{n-1}{n}\}\right)$$

- *$(X_1', X_2', \ldots, X_n')$ is still an element of Λ_3.*
- *$E(X_1, X_2, \ldots, X_n) \leq E(X_1', X_2', \ldots, X_n')$.*

By Proposition 3, we can further assume that $X_i^* \geq \frac{n-1}{n}$ for all i. That is, $(X_1^*, X_2^*, \ldots, X_n^*)$ has the following form:

$$(\frac{n-1}{n}, \frac{n-1}{n}, \ldots, \frac{n-1}{n}, [v], 1, 1, \ldots, 1, [w], X^*, X^*, \ldots, X^*)$$

In the above, $[v]$ represents that there is at most one value v in $(\frac{n-1}{n}, 1)$, and $[w]$ represents that there is at most one value w in $(1, X^*)$.

Proposition 4. *Let $(X_1, X_2, \ldots, X_n)$ be an element of Λ_3 with the following form*

$$(\frac{n-1}{n}, \frac{n-1}{n}, \ldots, \frac{n-1}{n}, [v], 1, 1, \ldots, 1, [w], X, X, \ldots, X)$$

In the above, $[v]$ represents that there is at most one value v in $(\frac{n-1}{n}, 1)$, and $[w]$ represents that there is at most one value w in $(1, X^*)$.

We use $(X_1', X_2', \ldots, X_n')$ to denote the following new element

$$(\max\{X_1, 1\}, \max\{X_2, 1\}, \ldots, \max\{X_n, 1\})$$

If $X_n \geq 1$, then

- *$(X_1', X_2', \ldots, X_n')$ is still an element of Λ_3.*
- *$E(X_1, X_2, \ldots, X_n) \leq E(X_1', X_2', \ldots, X_n') + \frac{4(n+1)^3}{27n(n-1)^2}$.*

Proof. We first prove that $(X'_1, X'_2, \ldots, X'_n)$ is still an element of Λ_3. By definition, $X = \frac{\sum_i X_i}{n-1} \geq 1$. We use X' to denote $\frac{\sum_i X'_i}{n-1}$. Certainly, $X' \geq X \geq 1$. Since $X \geq 1$ and $X \geq X_i$ for all i, we have $X' \geq X \geq X'_i = \max\{X_i, 1\} \geq 0$ for all i. Hence, $(X'_1, X'_2, \ldots, X'_n)$ is still in Λ_3.

Next, we compare $E(X_1, X_2, \ldots, X_n)$ and $E(X'_1, X'_2, \ldots, X'_n)$ under the assumption that $X_n \geq 1$.

If all the X_i are at least 1, then $E(X'_1, X'_2, \ldots, X'_n)$ equals $E(X_1, X_2, \ldots, X_n)$.

Next, we consider scenarios in which at least some X_i are less than 1. Let p be the highest index of X_i so that $X_p < 1$. Let $\Delta = X' - X$. Since $X_n \geq 1$, $\Delta = \frac{\sum_{i<n}(X'_i - X_i)}{n-1} \leq \frac{(n-1)\frac{1}{n}}{n-1} = \frac{1}{n}$. $E(X'_1, X'_2, \ldots, X'_n) - E(X_1, X_2, \ldots, X_n)$ simplifies to[7]

$$(n-1)\Delta + \frac{1}{n-1}\sum_i \sum_{j \neq i} \Bigg(\max\{X'_i + X'_j - X', \frac{n-1}{n}\}$$

$$- \max\{X_i + X_j - X, \frac{n-1}{n}\} \Bigg) - (n-1)\Delta$$

$$= \frac{1}{n-1}\sum_i \sum_{j \neq i} \left(\max\{X'_i + X'_j - X', \frac{n-1}{n}\} - \max\{X_i + X_j - X, \frac{n-1}{n}\} \right)$$

If $i < p$, then $X_i = \frac{n-1}{n}$. In this case, we have $X'_i + X'_j \geq (X_i + \frac{1}{n}) + X_j$. We also have $\Delta = X' - X \leq \frac{1}{n}$. Therefore, if $i < p$,

$$\max\{X'_i + X'_j - X', \frac{n-1}{n}\} - \max\{X_i + X_j - X, \frac{n-1}{n}\} \geq 0$$

Similarly, if $j < p$, we also have the above.

If $i \geq p$ and $j \geq p$, we have $X'_i + X'_j \geq X_i + X_j$. We also have $\Delta = X' - X = \frac{\sum_{i \leq p}(X'_i - X_i)}{n-1} \leq \frac{p}{n(n-1)}$. Therefore, if $i \geq p$ and $j \geq p$,

$$\max\{X'_i + X'_j - X', \frac{n-1}{n}\} - \max\{X_i + X_j - X, \frac{n-1}{n}\} \geq -\frac{p}{n(n-1)}$$

Based on the above,

$$E(X'_1, X'_2, \ldots, X'_n) - E(X_1, X_2, \ldots, X_n)$$

$$= \frac{1}{n-1}\sum_i \sum_{j \neq i} \left(\max\{X'_i + X'_j - X', \frac{n-1}{n}\} - \max\{X_i + X_j - X, \frac{n-1}{n}\} \right)$$

$$\geq \frac{1}{n-1}\sum_{i \geq p} \sum_{j \geq p, j \neq i} \left(\max\{X'_i + X'_j - X', \frac{n-1}{n}\} - \max\{X_i + X_j - X, \frac{n-1}{n}\} \right)$$

$$\geq \frac{1}{n-1}(n-p+1)(n-p)(-\frac{p}{n(n-1)})$$

[7] The fourth term of E stays the same.

$$\geq -\frac{1}{n(n-1)^2}(n-p+1)^2 p$$

$$\frac{\partial (n-p+1)^2 p}{\partial p} = (1+n-p)(1+n-3p)$$

Since $0 \leq p \leq n$, the maximum of $(n-p+1)^2 p$ can only happen when $p = 0$, $p = n$, or $p = \frac{n+1}{3}$. Hence,

$$(n-p+1)^2 p \leq \max\{0, n, \frac{4(n+1)^3}{27}\} = \frac{4(n+1)^3}{27}$$

Therefore,

$$E(X_1', X_2', \ldots, X_n') - E(X_1, X_2, \ldots, X_n) \geq -\frac{4(n+1)^3}{27n(n-1)^2}$$

Proposition 4 can help us further simplify the optimization problem. However, for it to apply, we need $X_n \geq 1$. If $X_n < 1$, then $(X_1^*, X_2^*, \ldots, X_n^*)$ has the following form:

$$(\frac{n-1}{n}, \frac{n-1}{n}, \ldots, \frac{n-1}{n}, [v])$$

E simplifies to

$$(n-1)X^* + \frac{1}{n-1}\sum_i \sum_{j \neq i} \frac{n-1}{n} - (n-1)X^* - \frac{n-2}{n-1}\sum_i 1 = \frac{1}{n-1}$$

We then consider cases where $X_n \geq 1$. Here, Proposition 4 does apply. Proposition 4 basically says that we can focus on the following form, and the resulting maximum plus $\frac{4(n+1)^3}{27n(n-1)^2}$ must be higher than or equal to the actual maximum.

$$(1, 1, \ldots, 1, [w], X^*, X^*, \ldots, X^*)$$

In the above, $[w]$ represents that there is at most one value w in $(1, X^*)$.

We allow w to be equal to 1 or X^*. The above form simplifies to

$$(1, 1, \ldots, 1, w, X^*, X^*, \ldots, X^*)$$

Let p be the number of 1s. To simplify E, we need to consider three separate cases:

- Case $1 + 1 - X^* \geq \frac{n-1}{n}$: E simplifies to[8]

$$\frac{1}{n-1}\sum_i \sum_{j \neq i}(X_i^* + X_j^* - X^*) - \frac{n-2}{n-1}(n-1)X^*$$

$$= \frac{1}{n-1}(2(n-1)^2 X^* - n(n-1)X^*) - \frac{n-2}{n-1}(n-1)X^*$$

$$= (2(n-1)X^* - nX^*) - (n-2)X^* = 0$$

[8] Term one and three cancel out.

– Case $1 + 1 - X^* < \frac{n-1}{n}$ and $1 + w - X^* \geq \frac{n-1}{n}$: E simplifies to

$$\frac{1}{n-1}\sum_i \sum_{j \neq i}(X_i^* + X_j^* - X^*) + \frac{1}{n-1}p(p-1)(\frac{n-1}{n} - 1 - 1 + X^*) - \frac{n-2}{n-1}(n-1)X^*$$

$$= \frac{1}{n-1}p(p-1)(\frac{n-1}{n} - 1 - 1 + X^*)$$

If $p = 0$ or 1, then the above expression is 0.
We then consider $p \geq 2$. To maximize the above, we want X^* to be as large as possible.

$$X^* = \frac{\sum_i X_i^*}{n-1} = \frac{p + w + (n-p-1)X^*}{n-1} \leq \frac{p + (n-p)X^*}{n-1}$$

$$(n-1)X^* \leq p + (n-p)X^*$$

$$(p-1)X^* \leq p$$

$$X^* \leq \frac{p}{p-1}$$

Hence, E is at most

$$\frac{1}{n-1}p(p-1)(\frac{n-1}{n} - 2 + \frac{p}{p-1})$$

$$= \frac{1}{n(n-1)}(-p^2 + p(n+1))$$

This expression is maximized when $p = \frac{n+1}{2}$. Hence,

$$E \leq \frac{(n+1)^2}{4n(n-1)} = \frac{1}{n-1} + \frac{n-1}{4n}$$

– Case $1 + w - X^* < \frac{n-1}{n}$: E simplifies to

$$\frac{1}{n-1}p(p-1)(\frac{n-1}{n} - 1 - 1 + X^*) + \frac{1}{n-1}2p(\frac{n-1}{n} - 1 - w + X^*)$$

If $p = 0$, then the above expression equals 0. We then consider $p \geq 1$. Since $X^* = \frac{\sum_i X_i^*}{n-1}$, we have $(n-1)X^* = p + w + (n-p-1)X^*$. That is, $X^* = \frac{p+w}{p}$. E is then

$$\frac{1}{n-1}p(p-1)(\frac{n-1}{n} - 1 + \frac{w}{p}) + \frac{1}{n-1}2p(\frac{n-1}{n} - w + \frac{w}{p})$$

The above expression is linear in w and w's coefficient equals

$$\frac{1}{n-1}(p-1) - \frac{1}{n-1}2p + \frac{1}{n-1}2 = \frac{1}{n-1}(1-p) \leq 0$$

To maximize the above expression, we let $w = 1$. E now equals

$$\frac{1}{n-1}p(p-1)(\frac{-1}{n}+\frac{1}{p})+\frac{1}{n-1}2p(\frac{-1}{n}+\frac{1}{p})$$
$$=\frac{1}{n(n-1)}(-p^2+p(n-1)+n)$$

The above expression is maximized when $p = \frac{n-1}{2}$. Hence, E is at most

$$\frac{1}{n(n-1)}(\frac{(n-1)^2}{4}+n)=\frac{n-1}{4n}+\frac{1}{n-1}$$

In conclusion, $\max_{\Lambda_3} E \leq \frac{1}{n-1}+\frac{n-1}{4n}+\frac{4(n+1)^3}{27n(n-1)^2}$.

Summary on the upper bound of *E*

$$\max_{\Lambda} E = \max\{\max_{\Lambda_1} E, \max_{\Lambda_2} E, \max_{\Lambda_3} E\}$$
$$\leq \frac{1}{n-1}+\frac{n-1}{4n}+\frac{4(n+1)^3}{27n(n-1)^2}$$

When n approaches infinity, this upper bound approaches $\frac{1}{4}+\frac{4}{27}\approx 0.398$.

4.2 Lower Bound of *E*

The process of finding a lower bound of E is similar.

$$\min_{\Lambda} E = \min\{\min_{\Lambda_1} E, \min_{\Lambda_2} E, \min_{\Lambda_3} E\}$$

Due to space constraint, we omit the details. We first show that if $X < 1$, then there must exist one $X_i < \frac{n-1}{n}$. Increasing X_i will never increase E. Therefore, as long as $X < 1$, we can push up values that are less than $\frac{n-1}{n}$ among the X_i. If all the X_i are at least $\frac{n-1}{n}$, then $X \geq 1$. In summary, when it comes to calculating the minimum value. It is without loss of generality to only consider Λ_3. That is,

$$\min_{\Lambda} E = \min_{\Lambda_3} E$$

Value of $\min_{\Lambda_3}$ *E*. We now analyze $\min_{\Lambda_3} E$. Let $(X_1^*, X_2^*, \ldots, X_n^*)$ be an element in Λ_3 that minimizes E. Let $X^* = \frac{\sum_i X_i^*}{n-1}$. Since $(X_1^*, X_2^*, \ldots, X_n^*) \in \Lambda_3$, $X^* \geq 1$. By symmetry, it is without loss of generality to assume that $X_1^* \leq X_2^* \leq \ldots \leq X_n^*$.

Since $1 \leq X^*$, the X_i^* fall into three possible bands. They are $[0, \frac{n-1}{n}]$, $[\frac{n-1}{n}, 1]$, and $[1, X^*]$. (Some bands may be empty.) By Proposition 2, it is without loss of generality to assume that values inside the same band are all identical. Hence, it is without loss of generality to assume that $(X_1^*, X_2^*, \ldots, X_n^*)$ has the following form:

$$(u, u, \ldots, u, v, v, \ldots, v, w, w, \ldots, w)$$

Here, $0 \le u \le \frac{n-1}{n}$, $\frac{n-1}{n} \le v \le 1$, and $1 \le w \le X^*$.

We have

$$\min_{\Lambda} E = \min_{\Lambda_3} E \ge \min\{\frac{1}{n-1} - \frac{1}{n} - \frac{(n-1)^2}{4n^2}, \frac{1}{n-1} + \frac{1}{2n} - \frac{1}{2}\} - \frac{n-2}{n(n-1)}$$

When n approaches infinity, this lower bound approaches $-\frac{1}{2} = -0.5$.

5 Proof of Theorem 1

If we redistribute according to the r_i', then the corresponding VCG redistribution mechanism must be non-deficit for the following reason (based on Proposition 1):

$$\sum_i r_i'(\theta_{-i}) = \sum_i r_i(\theta_{-i}) - U(n) \le VCG(\theta)$$

The achieved social welfare equals

$$\max\{\sum_i \theta_i, 1\} - VCG(\theta) + \sum_i r_i'(\theta_{-i})$$

$$= \max\{\sum_i \theta_i, 1\} - VCG(\theta) + \sum_i r_i(\theta_{-i}) - U(n)$$

$$\ge \max\{\sum_i \theta_i, 1\} + L(n) - U(n)$$

The worst-case efficiency ratio is then at least

$$\min_{\theta} \frac{\max\{\sum_i \theta_i, 1\} + L(n) - U(n)}{\max\{\sum_i \theta_i, 1\}} = 1 + \min_{\theta} \frac{L(n) - U(n)}{\max\{\sum_i \theta_i, 1\}} \ge 1 + L(n) - U(n) \quad (6)$$

We have the analytical forms of $L(n)$ and $U(n)$. When n goes to infinity, $1 + L(n) - U(n) = 1 - 0.5 - 0.398 = 0.102$. Actually, it is easy to verify that $1 + L(n) - U(n)$ is bounded below by a positive constant if $n > 10$. For $n \le 10$, based on the proof of Proposition 1, we know that the profiles that maximize/minimize E can only take a few specific forms. By numerically going over these forms (since $n \le 10$, it is computationally easy to do so), we can find the numerical values of $\max_{\Lambda} E$ and $\min_{\Lambda} E$ for $n \le 10$. Given a specific $n \le 10$, we use the numerical values to replace $U(n)$ and $L(n)$ in Eq. 6, which actually shows that the worst-case efficiency ratio is always bounded below by a positive constant.

6 Conclusion

In this paper, we proposed the first competitive VCG redistribution mechanism outside of the scope of resource allocation. The proposed mechanism is efficient, strategy-proof, non-deficit, and its social welfare is guaranteed to be at least a constant fraction of the optimal social welfare.

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