Vector Algebra and Calculus

1. Revision of vector algebra, scalar product, vector product
2. Triple products, multiple products, applications to geometry
3. Differentiation of vector functions, applications to mechanics
4. Scalar and vector fields. Line, surface and volume integrals, curvilinear co-ordinates
5. Vector operators — grad, div and curl
6. Vector Identities, curvilinear co-ordinate systems
7. **Gauss' and Stokes' Theorems, and extensions**
8. Engineering Applications

7. Gauss' and Stokes' Theorems

- This lecture finally begins to deliver on why we introduced div, grad and curl by introducing ...
- **Gauss' Theorem** enables an integral taken over a volume to be replaced by one taken over the surface bounding that volume, and vice versa.

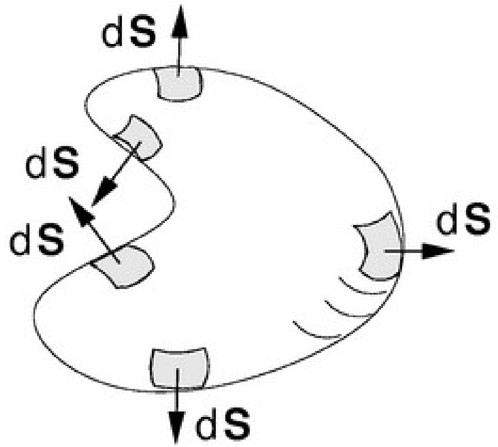
  Why would we want to do that?

  Computational efficiency and/or numerical accuracy come to mind.
- **Stokes' Theorem** enables an integral taken around a closed curve to be replaced by one taken over *any* surface bounded by that curve.

- Suppose that $\mathbf{a}(\mathbf{r})$ is a vector field, and we want to compute the **total flux of the field across the surface** S **that bounds a volume** V:

$$\int_S \mathbf{a} \cdot d\mathbf{S}$$

- $d\mathbf{S}$ is
  1. normal to the local surface element
  2. must everywhere point out of the volume



- Gauss' Theorem tells us that we can do this by considering the total flux generated inside the volume V:

Gauss' Theorem:

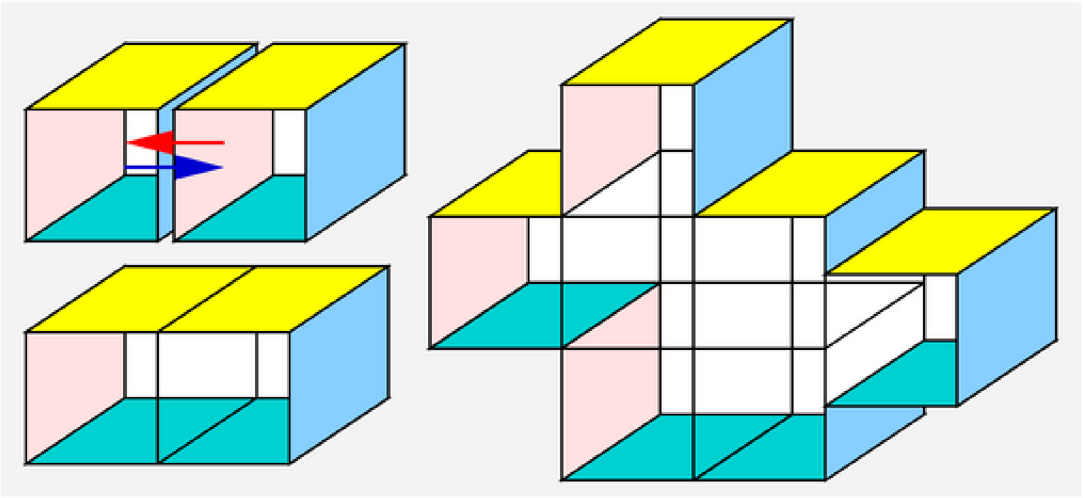
$$\int_S \mathbf{a} \cdot d\mathbf{S} = \int_V \operatorname{div} \mathbf{a} \;\; dV$$

Informal proof

- Divergence was *defined* as

$$\operatorname{div} \mathbf{a}\, dV = d(Efflux) = \sum_{\text{surface of } dV} \mathbf{a} \cdot d\mathbf{S} \,.$$

- If we sum over the volume elements, this results in a sum over the surface elements.

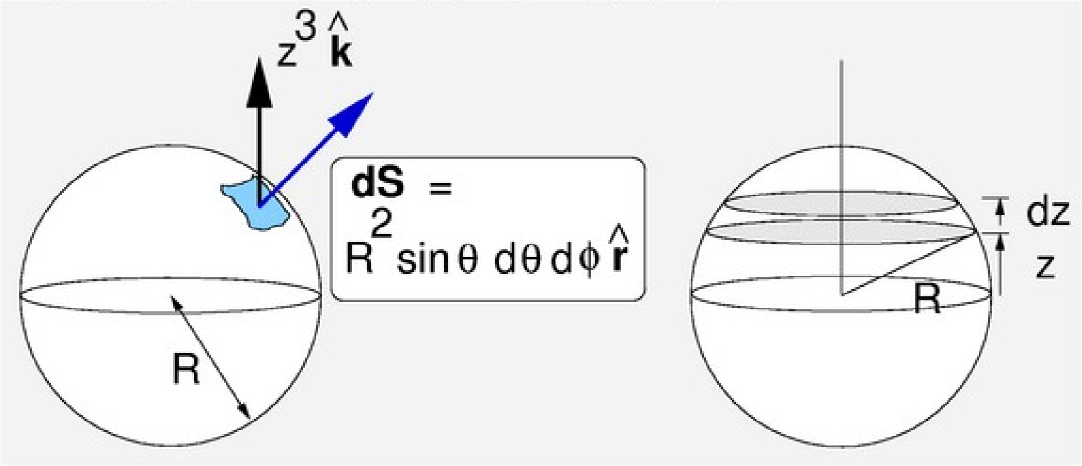


- But if two elemental surface touch, their $d\mathbf{S}$ vectors are in opposing direction and cancel.
- Thus the sum over surface elements gives the overall **bounding surface**.

$$\int_V \operatorname{div} \mathbf{a} \; dV = \int_{\text{Surface of } V} \mathbf{a} \cdot d\mathbf{S}$$

♣ Example of Gauss' Theorem

Q: Derive $\int_S \mathbf{a} \cdot d\mathbf{S}$ where $\mathbf{a} = z^3\hat{\mathbf{k}}$ and S is the surface of a sphere of radius R centred on the origin: (i) directly; (ii) by applying Gauss' Theorem.

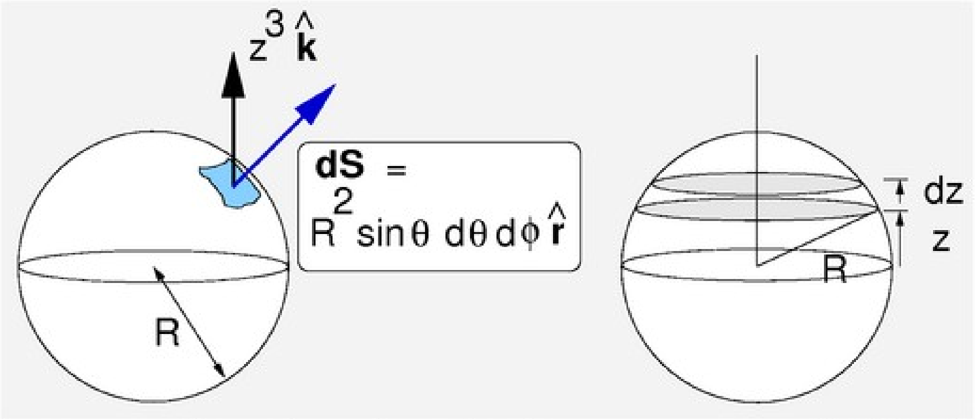


A(i): On the surface of the sphere, $\mathbf{a} = R^3 \cos^3\theta\hat{\mathbf{k}}$ and $d\mathbf{S} = R^2 \sin\theta d\theta d\phi\hat{\mathbf{r}}$. Everywhere $\hat{\mathbf{r}} \cdot \hat{\mathbf{k}} = \cos\theta$. Hence

$$\begin{aligned}\int_S \mathbf{a} \cdot d\mathbf{S} &= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} R^3\cos^3\theta \ . \ R^2\sin\theta d\theta d\phi \ . \ \cos\theta \\ &= 2\pi R^5 \int_0^{\pi} \cos^4\theta \sin\theta d\theta = \frac{2\pi R^5}{5}\left[-\cos^5\theta\right]_0^{\pi} = \frac{4\pi R^5}{5}\end{aligned}$$

A(ii): To apply Gauss' Theorem, we need (i) to work out div **a**

$$\mathbf{a} = z^3\hat{\mathbf{k}}, \Rightarrow \text{div}\mathbf{a} = 3z^2$$



(ii) Perform the volume integral. Because div**a** involves just z, we can divide the sphere into discs of constant z and thickness dz. Then

$$dV = \pi(R^2 - z^2)dz$$

So:

$$\int_V \text{div } \mathbf{a} dV = 3\pi \int_{-R}^{R} z^2(R^2 - z^2)dz$$

$$= 3\pi \left[\frac{R^2z^3}{3} - \frac{z^5}{5}\right]_{-R}^{R} = \frac{4\pi R^5}{5}$$

Typical: the surface integral is tedious, but volume integral is "straightforward" ...

An Extension to Gauss' Theorem

- Suppose vector field is $\mathbf{a} = U(\mathbf{r})\mathbf{c}$ with $U(\mathbf{r})$ a scalar field & $\mathbf{c}$ a **constant** vector. From Lecture 6 result and noting that $\text{div}\mathbf{c} = 0$:

$$\text{div } \mathbf{a} = \text{grad}U \cdot \mathbf{c} + U\text{div } \mathbf{c} \quad = \text{grad}U \cdot \mathbf{c}$$

- Gauss' Theorem tells us that

$$\int_S U\mathbf{c} \cdot d\mathbf{S} = \int_V \text{grad } U \cdot \mathbf{c} dV$$

But taking constant $\mathbf{c}$ outside ...

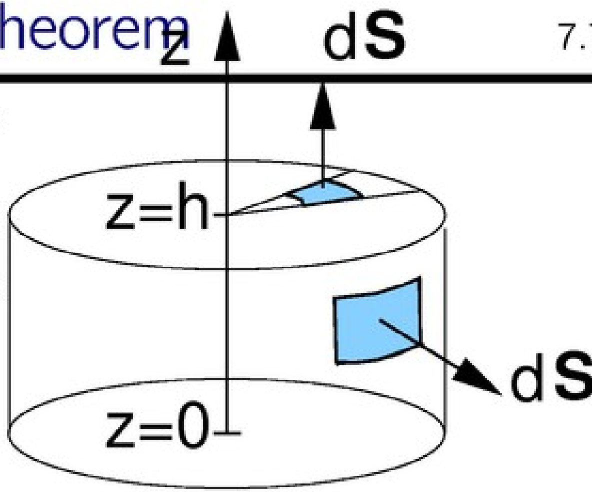
$$\mathbf{c} \cdot \left(\int_S U d\mathbf{S} \right) = \mathbf{c} \cdot \left(\int_V \text{grad } U dV \right)$$

- Now $\mathbf{c}$ is arbitrary so result must hold for any vector $\mathbf{c}$.

Hence, this Gauss-Theorem extension:

$$\int_S U d\mathbf{S} = \int_V \text{grad } U dV$$

♣ Example using this extension to Gauss' Theorem



Q: $U = x^2 + y^2 + z^2$ is a scalar field, and volume V is the cylinder $x^2 + y^2 \leq a^2$, $0 \leq z \leq h$. Compute the surface integral $\int_S U d\mathbf{S}$ over the surface of the cylinder.

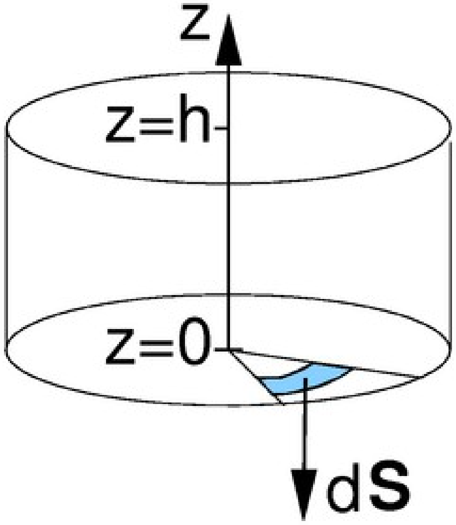
By direct surface integration ...

Symmetry gives zero contribution from curved surface, leaving

Top face:

$$U = (x^2 + y^2 + z^2) = (r^2 + h^2) \quad \text{and} \quad d\mathbf{S} = r\,dr\,d\phi\hat{\mathbf{k}}$$

$$\begin{aligned} \Rightarrow \int U d\mathbf{S} &= \int_{r=0}^{a} (h^2 + r^2) r\,dr \int_{\phi=0}^{2\pi} d\phi\hat{\mathbf{k}} \\ &= \left[\frac{1}{2}h^2 r^2 + \frac{1}{4}r^4\right]_0^a 2\pi\hat{\mathbf{k}} = \pi[h^2 a^2 + \frac{1}{2}a^4]\hat{\mathbf{k}} \end{aligned}$$



Bottom face:

$$U = (x^2 + y^2 + z^2) = r^2 \quad \& \quad d\mathbf{S} = -r\,dr\,d\phi\,\hat{\mathbf{k}}$$

$$\int U d\mathbf{S} = -\int_{r=0}^{a} r^3 dr \int_{\phi=0}^{2\pi} d\phi\,\hat{\mathbf{k}} = -\frac{\pi a^4}{2}\hat{\mathbf{k}}$$

$\Rightarrow$ Total integral is $\quad \pi[h^2 a^2 + \frac{1}{2}a^4]\hat{\mathbf{k}} - \frac{1}{2}\pi a^4 \hat{\mathbf{k}} = \quad \underline{\pi h^2 a^2 \hat{\mathbf{k}}.}$

Example, ctd: the volume integration

To test the RHS of the extension $\int_S U d\mathbf{S} = \int_V \text{grad}\, U dV$ we have to compute

$$\int_V \text{grad}\, U dV$$

$$U = x^2 + y^2 + z^2 \quad \Rightarrow \text{grad}\, U = 2(x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}})$$

So the integral is:

$$\begin{aligned}
& 2\int_V (x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}) r\ dr\ dz\ d\phi \\
= \ & 2\int_{z=0}^{h}\int_{r=0}^{a}\int_{\phi=0}^{2\pi} (r\cos\phi\hat{\mathbf{\imath}} + r\sin\phi\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}) r\ dr\ dz\ d\phi \\
= \ & 0\hat{\mathbf{\imath}} + 0\hat{\mathbf{\jmath}} + 2\int_{z=0}^{h} z dz \int_{r=0}^{a} r\ dr \int_{\phi=0}^{2\pi} d\phi\hat{\mathbf{k}} \\
= \ & \underline{\pi a^2 h^2 \hat{\mathbf{k}}}
\end{aligned}$$

Further extension to Gauss' Theorem

- Further "extensions" can be devised ...
- For example, apply Gauss' theorem to

$$\mathbf{a}(\mathbf{r}) = \mathbf{b}(\mathbf{r}) \times \mathbf{c}$$

where **c** is a constant vector ...

... and see what happens.

Stokes' Theorem

- Stokes' Theorem relates a line integral around a closed path ...

  ... to a surface integral over what is called a *capping surface* of the path.

Stokes' Theorem:

$$\oint_C \mathbf{a} \cdot d\mathbf{r} = \int_S \text{curl } \mathbf{a} \cdot d\mathbf{S}$$

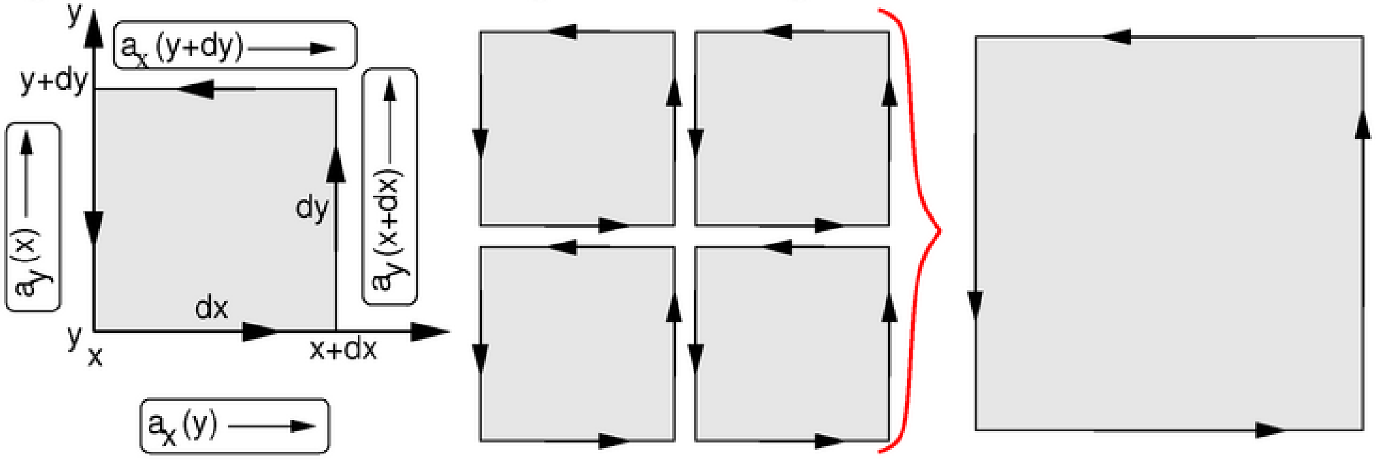
where S is *any* surface capping the curve C.

- Note, RHS is of course $\int (\text{curl } \mathbf{a}) \cdot d\mathbf{S}$. Why couldn't it be $\int \text{curl}(\mathbf{a} \cdot d\mathbf{S})$?

- Lecture 5 defined curl as the circulation per unit area, and showed that

$$\sum_{\text{around elemental loop}} \mathbf{a}\cdot d\mathbf{r} = dC = (\nabla\times\mathbf{a})\cdot d\mathbf{S} \quad .$$

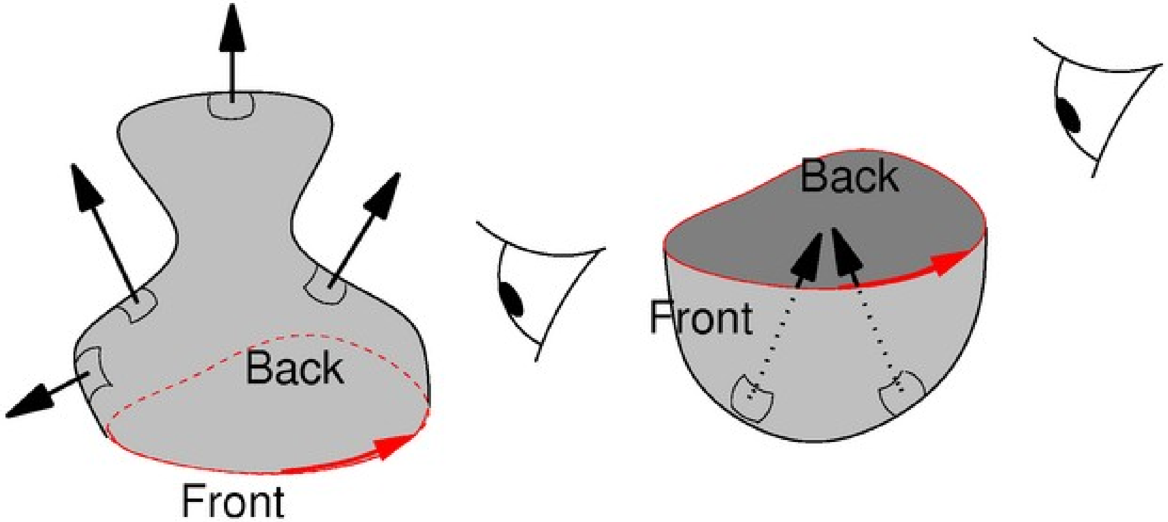
- If we add these little loops together, the internal line sections cancel out because the $d\mathbf{r}$'s are in opposite direction but the field $\mathbf{a}$ is not. This gives the larger surface and the larger bounding contour.



- In these diagrams the countour appears planar. In general the contour is a non-intersecting space curve.

Capping Surface

- The previous argument says that for a given contour, the capping surface can be **ANY** surface bound by the contour.
- Only requirement:
  the surface element vectors point in the "general direction" of a r-h screw w.r.t. to the sense of the contour integral.



- In practice, (in exam questions?!) the bounding contour is often planar, and the capping surface either flat, or hemispherical, or cylindrical.

♣ Example of Stokes' Theorem

Question: Vector field $\mathbf{a} = x^3\hat{\mathbf{j}} - y^3\hat{\mathbf{i}}$ and C is the circle of radius A centred on the origin.

Derive $\oint_C \mathbf{a} \cdot d\mathbf{r}$ (i) directly and (ii) using Stokes' with a planar surface.

Answer Direct: On the circle of radius A

$$\mathbf{a} = A^3(-\sin^3\theta\hat{\mathbf{i}} + \cos^3\theta\hat{\mathbf{j}})$$

and

$$d\mathbf{r} = Ad\theta(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}})$$

so that:

$$\oint_C \mathbf{a} \cdot d\mathbf{r} = \int_0^{2\pi} A^4(\sin^4\theta + \cos^4\theta)d\theta = \underline{\underline{\frac{3\pi}{2}A^4}},$$

since

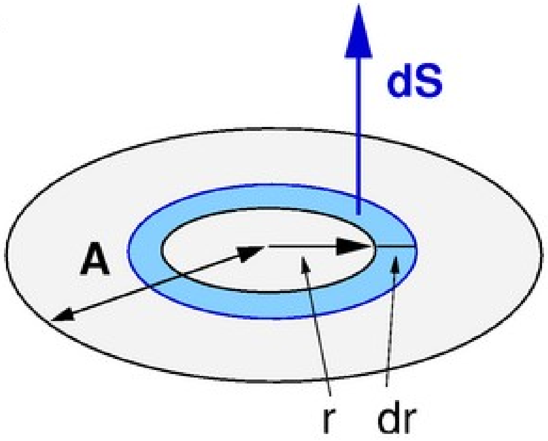
$$\int_0^{2\pi} \sin^4\theta d\theta = \int_0^{2\pi} \cos^4\theta d\theta = \frac{3\pi}{4}$$

Example /ctd

Answer Using Stokes' theorem $\int \text{curl}\mathbf{a} \cdot \ d\mathbf{S}$ over planar disc ...

$$\text{curl}\ \ \mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = 3(x^2 + y^2)\hat{\mathbf{k}} = 3r^2\hat{\mathbf{k}}$$

We choose area elements to be circular strips of radius r thickness dr. Then



$$d\mathbf{S} = r dr d\phi \hat{\mathbf{k}}$$

$$\int_S \text{curl}\mathbf{a} \cdot d\mathbf{S} = 3 \int_{\phi=0}^{2\pi} d\phi \int_{r=0}^{A} r^3 dr = \frac{3\pi}{2} A^4$$

An Extension to Stokes' Theorem

- Try similar "extension" with Stokes ...
- Again let $\mathbf{a}(\mathbf{r}) = U(\mathbf{r})\mathbf{c}$, where $\mathbf{c}$ is a constant vector. Then

$$\text{curl } \mathbf{a} = U \text{curl } \mathbf{c} + \text{grad } U \times \mathbf{c}$$

But curl $\mathbf{c}$ is zero. Stokes' Theorem becomes:

$$\oint_C U(\mathbf{c} \cdot d\mathbf{r}) = \int_S \text{grad } U \times \mathbf{c} \cdot d\mathbf{S} = \int_S \mathbf{c} \cdot (d\mathbf{S} \times \text{grad } U)$$

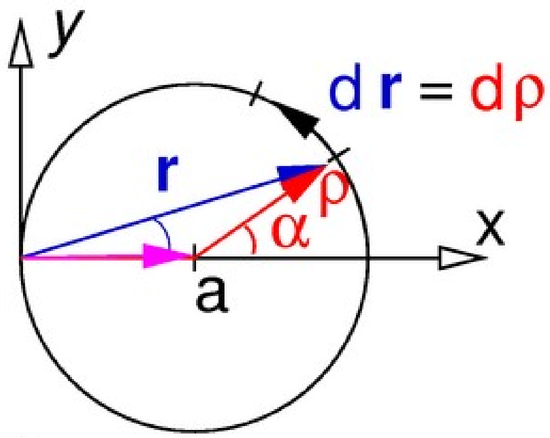
- Re-arranging and taking constant $\mathbf{c}$ out ...

$$\mathbf{c} \cdot \oint_C U d\mathbf{r} = -\mathbf{c} \cdot \int_S \text{grad} U \times d\mathbf{S} \quad .$$

But $\mathbf{c}$ is arbitrary and so

An extension to Stokes': $\oint_C U d\mathbf{r} = -\int_S \text{grad } U \times d\mathbf{S}$

♣ Example of extension to Stokes' Theorem



Question: Derive $\oint_C U d\mathbf{r}$ where $U = x^2 + y^2 + z^2$ and C is the circle $(x-a)^2 + y^2 = a^2, z = 0$, (i) directly and (ii) using Stokes with a planar capping surface.

A(i) Directly: On the circle $\mathbf{r} = a(1+\cos\alpha)\hat{\mathbf{i}} + a\sin\alpha\hat{\mathbf{j}}$, so

$$U = a^2(1+\cos\alpha)^2 + a^2\sin^2\alpha = 2a^2(1+\cos\alpha)$$

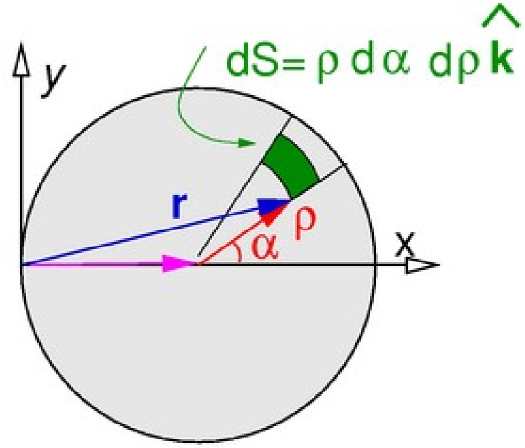
$$d\mathbf{r} = a\ d\alpha(-\sin\alpha\hat{\mathbf{i}} + \ \cos\alpha\hat{\mathbf{j}}) \quad .$$

So,

$$\oint U d\mathbf{r} = 2a^3 \int_{\alpha=0}^{2\pi} (1+\cos\alpha)(-\sin\alpha\hat{\mathbf{i}} + \cos\alpha\hat{\mathbf{j}})d\alpha = 2\pi a^3\hat{\mathbf{j}}$$

(It is worth checking that a zero $\hat{\mathbf{i}}$ component is indeed what you would expect from symmetry.)

A(ii) Using Stokes' ... Using the xy-planar surface



$$
\begin{aligned}
d\mathbf{S} &= \rho\, d\rho\, d\alpha\, \hat{\mathbf{k}} \\
\text{grad}\, U &= \text{grad} r^2 = 2\mathbf{r} \\
&= 2(a+\rho\cos\alpha)\hat{\mathbf{\imath}} + 2\rho\sin\alpha\hat{\mathbf{\jmath}} \quad ,
\end{aligned}
$$

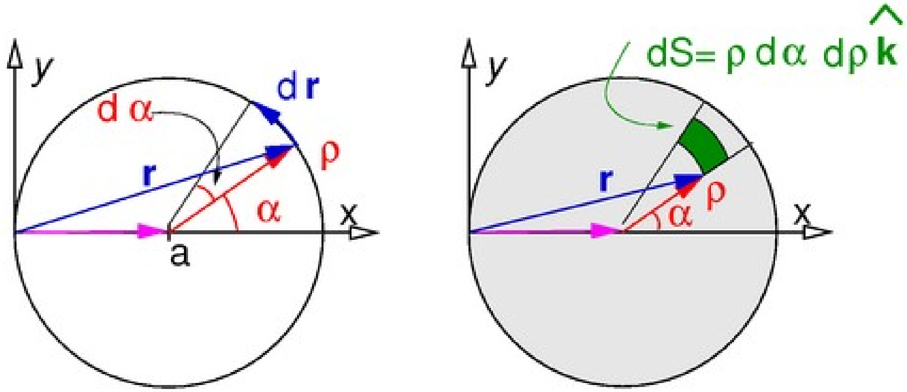
so that

$$
\begin{aligned}
d\mathbf{S} \times \text{grad}U &= \rho\, d\rho\, d\alpha[2(a+\rho\cos\alpha)(\hat{\mathbf{k}}\times\hat{\mathbf{\imath}}) + 2\rho\sin\alpha(\hat{\mathbf{k}}\times\hat{\mathbf{\jmath}})] \\
&= \rho\, d\rho\, d\alpha[-2\rho\sin\alpha\hat{\mathbf{\imath}} + 2(a+\rho\cos\alpha)\hat{\mathbf{\jmath}}]
\end{aligned}
$$

and as $\int_0^{2\pi}\sin\alpha\, d\alpha = 0$ and $\int_0^{2\pi}\cos\alpha\, d\alpha = 0$

$$
\begin{aligned}
\int_S d\mathbf{S} \times \text{grad}U &= \int_{\rho=0}^{a}\int_{\alpha=0}^{2\pi} \rho\, d\rho\, d\alpha (2a\hat{\mathbf{\jmath}}) \\
&= 2\pi\frac{a^2}{2}(2a\hat{\mathbf{\jmath}}) = 2\pi a^3\hat{\mathbf{\jmath}}
\end{aligned}
$$

I don't why understand why you used $\boldsymbol{\rho}$,α in the last example …



It is simply a coordinate transformation to decouple the coordinates. In the plane the general position is

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = r\cos\theta\hat{\mathbf{i}} + r\sin\theta\hat{\mathbf{j}} = (a + \rho\cos\alpha)\hat{\mathbf{i}} + +\rho\sin\alpha\hat{\mathbf{j}}$$

Going round the circumference, both r and θ change, so

$$d\mathbf{r} = (\cos\theta dr - r\sin\theta d\theta)\hat{\mathbf{i}} + (\sin\theta dr + r\cos\theta d\theta)\hat{\mathbf{j}}$$

whereas because $|\boldsymbol{\rho}| = a$ is constant

$$d\boldsymbol{\rho} = (-a\sin\alpha d\alpha)\hat{\mathbf{i}} + (a\cos\alpha d\alpha)\hat{\mathbf{j}}$$

Summary

In this lecture, we have developed

- **Gauss' Theorem**

$$\int_V \text{div } \mathbf{a} \;\; dV = \int_S \mathbf{a} \cdot d\mathbf{S}$$

  If you sum up the δ(Effluxes) from each δ(Volume) in an object, you must end up with the overall efflux from the surface.

- **Stokes' Theorem**

$$\oint_C \mathbf{a} \cdot d\mathbf{r} = \int_S \text{curl } \mathbf{a} \cdot d\mathbf{S}$$

  which says if you add up the δ(Circulations) per unit area over an open surface, you end up with the Circulation around the rim

- We've seen how to verify and apply the theorems to simple surfaces and volumes using Cartesians, cylindrical and spherical polars.