Lecture 6

Vector Operator Identities

In this lecture we look at more complicated identities involving vector operators. The main thing to appreciate it that the operators behave both as vectors and as differential operators, so that the usual rules of taking the derivative of, say, a product must be observed.

There could be a cottage industry inventing vector identities. HLT contains a lot of them. So why not leave it at that?

First, since grad, div and curl describe key aspects of vectors fields, they arise often in practice, and so the identities can save you a lot of time and hacking of partial derivatives, as we will see when we consider Maxwell's equation as an example later.

Secondly, they help to identify other practically important vector operators. So, although this material is a bit dry, the relevance of the identities should become clear later in other Engineering courses.

6.1 Identity 1: curl grad $U = \mathbf{0}$

$$
\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{\nabla} U &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial U/\partial x & \partial U/\partial y & \partial U/\partial z \end{vmatrix} \\
&= \hat{\boldsymbol{\imath}} \left(\frac{\partial^2 U}{\partial y \partial z} - \frac{\partial^2 U}{\partial z \partial y} \right) + \hat{\boldsymbol{\jmath}}\,() + \hat{\boldsymbol{k}}\,() \\
&= \mathbf{0} \quad ,
\end{aligned}
$$

as $\partial^2/\partial y\partial z = \partial^2/\partial z\partial y$.

Note that the output is a null *vector*.

6.2 Identity 2: div curl $\mathbf{a} = 0$

$$\begin{aligned}
\nabla \cdot \nabla \times \mathbf{a} &= \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix} \\
&= \frac{\partial^2 a_z}{\partial x \partial y} - \frac{\partial^2 a_y}{\partial x \partial z} - \frac{\partial^2 a_z}{\partial y \partial x} + \frac{\partial^2 a_x}{\partial y \partial z} + \frac{\partial^2 a_y}{\partial z \partial x} - \frac{\partial^2 a_x}{\partial z \partial y} \\
&= 0
\end{aligned}$$

6.3 Identity 3: div and curl of $U\mathbf{a}$

Suppose that $U(\mathbf{r})$ is a scalar field and that $\mathbf{a}(\mathbf{r})$ is a vector field and we are interested in the product $U\mathbf{a}$. This is a vector field, so we can compute its divergence and curl. For example the density $\rho(\mathbf{r})$ of a fluid is a scalar field, and the instantaneous velocity of the fluid $\mathbf{v}(\mathbf{r})$ is a vector field, and we are probably interested in mass flow rates for which we will be interested in $\rho(\mathbf{r})\mathbf{v}(\mathbf{r})$.

The divergence (a scalar) of the product $U\mathbf{a}$ is given by:

$$\begin{aligned}
\nabla \cdot (U\mathbf{a}) &= U(\nabla \cdot \mathbf{a}) + (\nabla U) \cdot \mathbf{a} \\
&= U\mathrm{div}\mathbf{a} + (\mathrm{grad}U) \cdot \mathbf{a}
\end{aligned}$$

In a similar way, we can take the curl of the vector field $U\mathbf{a}$, and the result should be a vector field:

$$\nabla \times (U\mathbf{a}) = U\nabla \times \mathbf{a} + (\nabla U) \times \mathbf{a} \quad .$$

6.4 Identity 4: div of $\mathbf{a} \times \mathbf{b}$

Life quickly gets trickier when vector or scalar products are involved: For example, it is not *that* obvious that

$$\mathrm{div}(\mathbf{a} \times \mathbf{b}) = \mathrm{curl}\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathrm{curl}\mathbf{b}$$

To show this, use the determinant:

$$\begin{aligned}
\begin{vmatrix} \partial/\partial x_i & \partial/\partial x_j & \partial/\partial x_k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} &= \frac{\partial}{\partial x}[a_y b_z - a_z b_y] + \frac{\partial}{\partial y}[a_z b_x - a_x b_z] + \frac{\partial}{\partial z}[a_x b_y - a_y b_x] \\
&= \ldots \text{ bash out the products } \ldots \\
&= \mathrm{curl}\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot (\mathrm{curl}\ \mathbf{b})
\end{aligned}$$

6.5 Identity 5: curl($\mathbf{a} \times \mathbf{b}$)

$$\mathrm{curl}(\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_y b_z - a_z b_y & a_z b_x - a_x b_z & a_x b_y - a_y b_x \end{vmatrix}$$

so the $\hat{\boldsymbol{\imath}}$ component is

$$\frac{\partial}{\partial y}(a_x b_y - a_y b_x) - \frac{\partial}{\partial z}(a_z b_x - a_x b_z)$$

which can be written as the sum of four terms:

$$a_x \left(\frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z} \right) - b_x \left(\frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \right) + \left(b_y \frac{\partial}{\partial y} + b_z \frac{\partial}{\partial z} \right) a_x - \left(a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z} \right) b_x$$

Adding $a_x(\partial b_x/\partial x)$ to the first of these, and subtracting it from the last, and doing the same with $b_x(\partial a_x/\partial x)$ to the other two terms, we find that (you should of course check this):

$$\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{b}) = (\boldsymbol{\nabla} \cdot \mathbf{b})\mathbf{a} - (\boldsymbol{\nabla} \cdot \mathbf{a})\mathbf{b} + [\mathbf{b} \cdot \boldsymbol{\nabla}]\mathbf{a} - [\mathbf{a} \cdot \boldsymbol{\nabla}]\mathbf{b}$$

where $[\mathbf{a} \cdot \boldsymbol{\nabla}]$ can be regarded as new, and very useful, scalar differential operator.

6.6 Definition of the operator $[\mathbf{a} \cdot \boldsymbol{\nabla}]$

This is a *scalar operator*, but it can obviously can be applied to a scalar field, resulting in a scalar field, or to a vector field resulting in a vector field:

$$[\mathbf{a} \cdot \boldsymbol{\nabla}] \equiv \left[a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z} \right] \quad .$$

6.7 Identity 6: curl(curl$\mathbf{a}$) for you to derive

The following important identity is stated, and left as an exercise:

$$\mathrm{curl}(\mathrm{curl}\mathbf{a}) = \mathrm{graddiv}\mathbf{a} - \nabla^2 \mathbf{a}$$

where

$$\nabla^2 \mathbf{a} = \nabla^2 a_x \hat{\boldsymbol{\imath}} + \nabla^2 a_y \hat{\boldsymbol{\jmath}} + \nabla^2 a_z \hat{\boldsymbol{k}}$$

♣ Example of Identity 6: electromagnetic waves

Q: James Clerk Maxwell established a set of four vector equations which are fundamental to working out how eletromagnetic waves propagate. The entire telecommunications industry is built on these.

$$\begin{aligned}
\mathrm{div}\mathbf{D} &= \rho \\
\mathrm{div}\mathbf{B} &= 0 \\
\mathrm{curl}\mathbf{E} &= -\frac{\partial}{\partial t}\mathbf{B} \\
\mathrm{curl}\mathbf{H} &= \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D}
\end{aligned}$$

In addition, we can assume the following, which should all be familiar to you:
$\mathbf{B} = \mu_r\mu_0\mathbf{H}$, $\mathbf{J} = \sigma\mathbf{E}$, $\mathbf{D} = \epsilon_r\epsilon_0\mathbf{E}$,
where all the scalars are constants.

Now show that in a material with zero free charge density, $\rho = 0$, and with zero conductivity, $\sigma = 0$, the electric field $\mathbf{E}$ must be a solution of the wave equation

$$\nabla^2\mathbf{E} = \mu_r\mu_0\epsilon_r\epsilon_0(\partial^2\mathbf{E}/\partial t^2) \ .$$

A: First, a bit of respect. Imagine you are the first to do this — this is a tingle moment.

$$\begin{aligned}
\mathrm{div}\mathbf{D} &= \mathrm{div}(\epsilon_r\epsilon_0\mathbf{E}) = \epsilon_r\epsilon_0\mathrm{div}\mathbf{E} = \rho = 0 \quad \Rightarrow \mathrm{div}\mathbf{E} = 0. \quad (a) \\
\mathrm{div}\mathbf{B} &= \mathrm{div}(\mu_r\mu_0\mathbf{H}) = \mu_r\mu_0\mathrm{div}\mathbf{H} = 0 \quad \Rightarrow \mathrm{div}\mathbf{B} = 0 \quad (b) \\
\mathrm{curl}\mathbf{E} &= -\partial\mathbf{B}/\partial t = -\mu_r\mu_0(\partial\mathbf{H}/\partial t) \quad (c) \\
\mathrm{curl}\mathbf{H} &= \mathbf{J} + \partial\mathbf{D}/\partial t = \mathbf{0} + \epsilon_r\epsilon_0(\partial\mathbf{E}/\partial t) \quad (d)
\end{aligned}$$

But we know (or rather you worked out in Identity 6) that curlcurl = graddiv$-\nabla^2$, and using (c)

$$\mathrm{curlcurl}\mathbf{E} = \mathrm{graddiv}\mathbf{E} - \nabla^2\mathbf{E} = \mathrm{curl}\left(-\mu_r\mu_0(\partial\mathbf{H}/\partial t)\right)$$

so interchanging the order of partial differentation, and using (a) div$\mathbf{E} = 0$:

$$\begin{aligned}
-\nabla^2\mathbf{E} &= -\mu_r\mu_0\frac{\partial}{\partial t}(\mathrm{curl}\mathbf{H}) \\
&= -\mu_r\mu_0\frac{\partial}{\partial t}\left(\epsilon_r\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right) \\
\Rightarrow \nabla^2\mathbf{E} &= \mu_r\mu_0\epsilon_r\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}
\end{aligned}$$

This equation is actually three equations, one for each component:

$$\nabla^2 E_x = \mu_r \mu_0 \epsilon_r \epsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

and so on for E_y and E_z.

6.8 Grad, div, curl and ∇^2 in curvilinear co-ordinate systems

It is possible to obtain general expressions for grad, div and curl in any orthogonal curvilinear co-ordinate system by making use of the h factors which were introduced in Lecture 4.

We recall that the unit vector in the direction of increasing u, with v and w being kept constant, is

$$\hat{\mathbf{u}} = \frac{1}{h_u}\frac{\partial \mathbf{r}}{\partial u}$$

where $\mathbf{r}$ is the position vector, and

$$h_u = \left|\frac{\partial \mathbf{r}}{\partial u}\right|$$

is the metric coefficient. Similar expressions apply for the other co-ordinate directions. Then

$$d\mathbf{r} = h_u \hat{\mathbf{u}} du + h_v \hat{\mathbf{v}} dv + h_w \hat{\mathbf{w}} dw \ .$$

6.9 Grad in curvilinear coordinates

Noting that $U = U(\mathbf{r})$ and $U = U(u, v, w)$, and using the properties of the gradient of a scalar field obtained previously

$$\nabla U \cdot d\mathbf{r} = dU = \frac{\partial U}{\partial u} du + \frac{\partial U}{\partial v} dv + \frac{\partial U}{\partial w} dw$$

It follows that

$$\nabla U \cdot (h_u \hat{\mathbf{u}} du + h_v \hat{\mathbf{v}} dv + h_w \hat{\mathbf{w}} dw) = \frac{\partial U}{\partial u} du + \frac{\partial U}{\partial v} dv + \frac{\partial U}{\partial w} dw$$

The only way this can be satisfied for independent du, dv, dw is when

$$\boxed{\nabla U = \frac{1}{h_u}\frac{\partial U}{\partial u}\hat{\mathbf{u}} + \frac{1}{h_v}\frac{\partial U}{\partial v}\hat{\mathbf{v}} + \frac{1}{h_w}\frac{\partial U}{\partial w}\hat{\mathbf{w}}}$$

6.10 Divergence in curvilinear coordinates

Expressions can be obtained for the divergence of a vector field in orthogonal curvilinear co-ordinates by making use of the flux property.

We consider an element of volume dV. If the curvilinear coordinates are orthogonal then the little volume is a cuboid (to first order in small quantities) and

$$dV = h_u\ h_v\ h_w\ \ du\ dv\ dw\quad .$$

However, it is not quite a cuboid: the area of two opposite faces will differ as the scale parameters are functions of u, v and w in general.

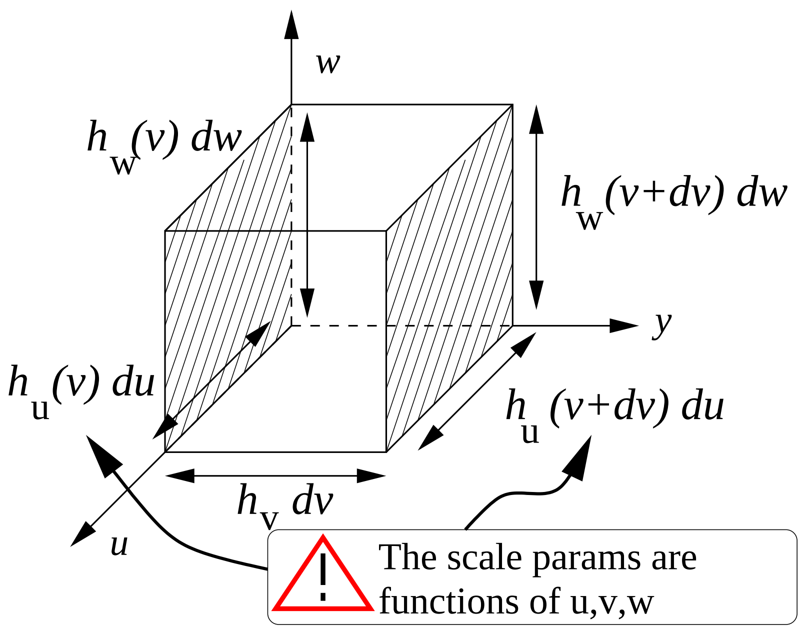


Figure 6.1: Elemental volume for calculating divergence in orthogonal curvilinear coordinates

So the net efflux from the two faces in the $\hat{\mathbf{v}}$ direction shown in Figure 6.1 is

$$\begin{aligned} &= \left[a_v + \frac{\partial a_v}{\partial v}dv\right]\left[h_u + \frac{\partial h_u}{\partial v}dv\right]\left[h_w + \frac{\partial h_w}{\partial v}dv\right]dudw - a_v h_u h_w dudw \\ &= \frac{\partial(a_v h_u h_w)}{\partial v}dudvdw \end{aligned}$$

which is easily shown by multiplying the first line out and dropping second order terms (i.e. $(dv)^2$).

By definition div is the net efflux per unit volume, so summing up the other faces:

$$\begin{aligned} \text{div}\mathbf{a}\ dV &= \left(\frac{\partial(a_u\ h_v\ h_w)}{\partial u} + \frac{\partial(a_v\ h_u\ h_w)}{\partial v} + \frac{\partial(a_w\ h_u\ h_v)}{\partial w}\right) dudvdw \\ \Rightarrow \text{div}\mathbf{a}\ h_u h_v h_w\ dudvdw &= \left(\frac{\partial(a_u\ h_v\ h_w)}{\partial u} + \frac{\partial(a_v\ h_u\ h_w)}{\partial v} + \frac{\partial(a_w\ h_u\ h_v)}{\partial w}\right) dudvdw \end{aligned}$$

So, finally,

$$\boxed{\operatorname{div}\mathbf{a} = \frac{1}{h_u h_v h_w}\left(\frac{\partial(a_u\ h_v\ h_w)}{\partial u} + \frac{\partial(a_v\ h_u\ h_w)}{\partial v} + \frac{\partial(a_w\ h_u\ h_v)}{\partial w}\right)}$$

6.11 Curl in curvilinear coordinates

Recall from Lecture 5 that we computed the z component of curl as the circulation per unit area from

$$dC = \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\ dx\ dy$$

By analogy with our derivation of divergence, you will realize that for an orthogonal curvilinear coordinate system we can write the area as $h_u h_v du dw$. But the opposite sides are no longer quite of the same length. The lower of the pair in Figure 6.2 is length $h_u(v)du$, but the upper is of length $h_u(v+dv)du$

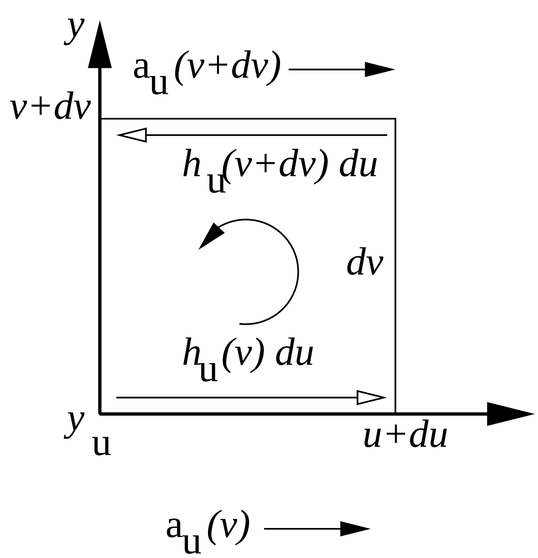


Figure 6.2: Elemental loop for calculating curl in orthogonal curvilinear coordinates

Summing this pair gives a contribution to the circulation

$$a_u(v)h_u(v)du - a_u(v+dv)h_u(v+dv)du = -\frac{\partial(h_u a_u)}{\partial v}dvdu$$

and together with the other pair:

$$dC = \left(-\frac{\partial(h_u a_u)}{\partial v} + \frac{\partial(h_v a_v)}{\partial u}\right)dudv$$

So the circulation per unit area is

$$\frac{dC}{h_u h_v du dv} = \frac{1}{h_u h_v}\left(\frac{\partial(h_v a_v)}{\partial u} - \frac{\partial(h_u a_u)}{\partial v}\right)$$

and hence curl is

$$\begin{aligned}\mathrm{curl}\mathbf{a}(u,v,w) \;=\; & \frac{1}{h_v h_w}\left(\frac{\partial(h_w a_w)}{\partial v} - \frac{\partial(h_v a_v)}{\partial w}\right)\hat{\mathbf{u}} \;+ \\ & \frac{1}{h_w h_u}\left(\frac{\partial(h_u a_u)}{\partial w} - \frac{\partial(h_w a_w)}{\partial u}\right)\hat{\mathbf{v}} \;+ \\ & \frac{1}{h_u h_v}\left(\frac{\partial(h_v a_v)}{\partial u} - \frac{\partial(h_u a_u)}{\partial v}\right)\hat{\mathbf{w}}\end{aligned}$$

You should check that this can be written as

Curl in curvilinear coords:

$$\mathrm{curl}\mathbf{a}(u,v,w) \;=\; \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{\mathbf{u}} & h_v\hat{\mathbf{v}} & h_w\hat{\mathbf{w}} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u a_u & h_v a_v & h_w a_w \end{vmatrix}$$

6.12 The Laplacian in curvilinear coordinates

Substitution of the components of grad$U$ into the expression for div$\mathbf{a}$ immediately (!*?) gives the following expression for the Laplacian in general orthogonal co-ordinates:

$$\nabla^2 U = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial U}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_w h_u}{h_v}\frac{\partial U}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial U}{\partial w}\right)\right].$$

6.13 Grad Div, Curl, ∇^2 in cylindrical polars

Here $(u,v,w) \to (r,\phi,z)$. The position vector is $\mathbf{r} = r\cos\phi\hat{\imath} + r\sin\phi\hat{\jmath} + z\hat{\boldsymbol{k}}$, and $h_r = |\partial\mathbf{r}/\partial r|$, etc.

$$\begin{aligned}\Rightarrow h_r &= \sqrt{(\cos^2\phi + sin^2\phi)} = 1, \\ h_\phi &= \sqrt{(r^2\sin^2\phi + r^2\cos^2\phi)} = r, \\ h_z &= 1\end{aligned}$$

$$
\begin{aligned}
\Rightarrow \text{grad}U &= \frac{\partial U}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\mathbf{e}}_\phi + \frac{\partial U}{\partial z}\hat{\boldsymbol{k}} \\
\text{div}\mathbf{a} &= \frac{1}{r}\left(\frac{\partial (r a_r)}{\partial r} + \frac{\partial a_\phi}{\partial \phi}\right) + \frac{\partial a_z}{\partial z} \\
\text{curl}\mathbf{a} &= \left(\frac{1}{r}\frac{\partial a_z}{\partial \phi} - \frac{\partial a_\phi}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)\hat{\mathbf{e}}_\phi + \frac{1}{r}\left(\frac{\partial (r a_\phi)}{\partial r} - \frac{\partial a_r}{\partial \phi}\right)\hat{\boldsymbol{k}} \\
\nabla^2 U &= \text{Tutorial Exercise}
\end{aligned}
$$

6.14 Grad Div, Curl, ∇^2 in spherical polars

Here $(u, v, w) \to (r, \theta, \phi)$. The position vector is $\mathbf{r} = r\sin\theta\cos\phi\hat{\boldsymbol{\imath}} + r\sin\theta\sin\phi\hat{\boldsymbol{\jmath}} + r\cos\theta\hat{\boldsymbol{k}}$.

$$
\begin{aligned}
\Rightarrow h_r &= \sqrt{(\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta)} = 1 \\
h_\theta &= \sqrt{(r^2\cos^2\theta(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta)} = r \\
h_\phi &= \sqrt{(r^2\sin^2\theta(\sin^2\phi + \cos^2\phi)} = r\sin\theta
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \text{grad}U &= \frac{\partial U}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\mathbf{e}}_\phi \\
\text{div}\mathbf{a} &= \frac{1}{r^2}\frac{\partial (r^2 a_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (a_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial \phi} \\
\text{curl}\mathbf{a} &= \frac{\hat{\mathbf{e}}_r}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(a_\phi\sin\theta) - \frac{\partial}{\partial \phi}(a_\theta)\right) + \frac{\hat{\mathbf{e}}_\theta}{r\sin\theta}\left(\frac{\partial}{\partial \phi}(a_r) - \frac{\partial}{\partial r}(a_\phi r\sin\theta)\right) + \\
&\qquad \frac{\hat{\mathbf{e}}_\phi}{r}\left(\frac{\partial}{\partial r}(a_\theta r) - \frac{\partial}{\partial \theta}(a_r)\right) \\
\nabla^2 U &= \text{Tutorial Exercise}
\end{aligned}
$$

♣ Examples

Q1 Find curl$\mathbf{a}$ in (i) Cartesians and (ii) Spherical polars when $\mathbf{a} = x(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}})$.

A1 (i) In Cartesians

$$
\text{curl}\mathbf{a} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & xy & xz \end{vmatrix} = -z\hat{\boldsymbol{\jmath}} + y\hat{\boldsymbol{k}} \ .
$$

(ii) In spherical polars, $x = r\sin\theta\cos\phi$ and $\mathbf{r} = (x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}})$. So

$$\begin{aligned} \mathbf{a} &= r^2\sin\theta\cos\phi\hat{\mathbf{e}}_r \\ \Rightarrow a_r &= r^2\sin\theta\cos\phi; \qquad a_\theta = 0; \qquad a_\phi = 0 \,. \end{aligned}$$

Hence as

$$\mathrm{curl}\mathbf{a} = \frac{\hat{\mathbf{e}}_r}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(a_\phi\sin\theta) - \frac{\partial}{\partial\phi}(a_\theta)\right) + \frac{\hat{\mathbf{e}}_\theta}{r\sin\theta}\left(\frac{\partial}{\partial\phi}(a_r) - \frac{\partial}{\partial r}(a_\phi r\sin\theta)\right) + \frac{\hat{\mathbf{e}}_\phi}{r}\left(\frac{\partial}{\partial r}(a_\theta r) - \frac{\partial}{\partial\theta}(a_r)\right)$$

$$\begin{aligned} \mathrm{curl}\mathbf{a} &= \frac{\hat{\mathbf{e}}_\theta}{r\sin\theta}\left(\frac{\partial}{\partial\phi}(r^2\sin\theta\cos\phi)\right) + \frac{\hat{\mathbf{e}}_\phi}{r}\left(-\frac{\partial}{\partial\theta}(r^2\sin\theta\cos\phi)\right) \\ &= \frac{\hat{\mathbf{e}}_\theta}{r\sin\theta}(-r^2\sin\theta\sin\phi) + \frac{\hat{\mathbf{e}}_\phi}{r}\left(-r^2\cos\theta\cos\phi)\right) \\ &= \hat{\mathbf{e}}_\theta(-r\sin\phi) + \hat{\mathbf{e}}_\phi(-r\cos\theta\cos\phi) \end{aligned}$$

Checking: these two results should be the same, but to check we need expressions for $\hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi$ in terms of $\hat{\boldsymbol{\imath}}$ etc.

Remember that we can work out the unit vectors $\hat{\mathbf{e}}_r$ and so on in terms of $\hat{\boldsymbol{\imath}}$ etc using

$$\hat{\mathbf{e}}_r = \frac{1}{h_1}\frac{\partial\mathbf{r}}{dr}; \qquad \hat{\mathbf{e}}_\theta = \frac{1}{h_2}\frac{\partial\mathbf{r}}{d\theta}; \qquad \hat{\mathbf{e}}_\phi = \frac{1}{h_3}\frac{\partial\mathbf{r}}{d\phi} \quad \text{where } \mathbf{r} = x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}+z\hat{\boldsymbol{k}} \;.$$

Grinding through we find

$$\begin{bmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{k}} \end{bmatrix} = [R]\begin{bmatrix} \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{k}} \end{bmatrix}$$

Don't be shocked to see a rotation matrix $[R]$: we are after all rotating one right-handed orthogonal coord system into another.

So the result in spherical polars is

$$\begin{aligned} \mathrm{curl}\mathbf{a} &= (\cos\theta\cos\phi\hat{\boldsymbol{\imath}} + \cos\theta\sin\phi\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{k}})(-r\sin\phi) + (-\sin\phi\hat{\boldsymbol{\imath}} + \cos\phi\hat{\boldsymbol{\jmath}})(-r\cos\theta\cos\phi) \\ &= -r\cos\theta\hat{\boldsymbol{\jmath}} + r\sin\theta\sin\phi\hat{\boldsymbol{k}} \\ &= -z\hat{\boldsymbol{\jmath}} + y\hat{\boldsymbol{k}} \end{aligned}$$

which is exactly the result in Cartesians.

Q2 Find the divergence of the vector field $\mathbf{a} = r\mathbf{c}$ where $\mathbf{c}$ is a constant vector (i) using Cartesian coordinates and (ii) using Spherical Polar coordinates.

A2 (i) Using Cartesian coords:

$$\begin{aligned}
\mathrm{div}\mathbf{a} &= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}c_x + \ldots \\
&= x.(x^2+y^2+z^2)^{-1/2}c_x + \ldots \\
&= \frac{1}{r}\mathbf{r}\cdot\mathbf{c} \quad .
\end{aligned}$$

(ii) Using Spherical polars

$$\mathbf{a} = a_r\hat{\mathbf{e}}_r + a_\theta\hat{\mathbf{e}}_\theta + a_\phi\hat{\mathbf{e}}_\phi$$

and our first task is to find a_r and so on. We can't do this by inspection, and finding their values requires more work than you might think! Recall

$$\begin{bmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{k}} \end{bmatrix} \quad = \quad [R] \begin{bmatrix} \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{k}} \end{bmatrix}$$

Now the point is the same point in space whatever the coordinate system, so

$$a_r\hat{\mathbf{e}}_r + a_\theta\hat{\mathbf{e}}_\theta + a_\phi\hat{\mathbf{e}}_\phi = a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} + a_z\hat{\boldsymbol{k}}$$

and using the inner product

$$\begin{aligned}
\begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}^\top \begin{bmatrix} \hat{\mathbf{e}}_r \\ \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_\phi \end{bmatrix} &= \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}^\top \begin{bmatrix} \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{k}} \end{bmatrix} \\
\begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}^\top [R] \begin{bmatrix} \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{k}} \end{bmatrix} &= \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}^\top \begin{bmatrix} \hat{\boldsymbol{\imath}} \\ \hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{k}} \end{bmatrix} \\
\Rightarrow \begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}^\top [R] &= \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}^\top \\
\Rightarrow \begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix}^\top &= \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}^\top [R]^\top \\
\Rightarrow \begin{bmatrix} a_r \\ a_\theta \\ a_\phi \end{bmatrix} &= [R] \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}
\end{aligned}$$

For our particular problem, $a_x = rc_x$, etc, where c_x is a constant, so now we can write down

$$\begin{aligned} a_r &= r(\sin\theta\cos\phi c_x \;+\; \sin\theta\sin\phi c_y \;+\; \cos\theta c_z) \\ a_\theta &= r(\cos\theta\cos\phi c_x \;+\; \cos\theta\sin\phi c_y \;-\; \sin\theta c_z) \\ a_\phi &= r(-\sin\phi c_x \;+\; \cos\phi c_y) \end{aligned}$$

Now all we need to do is to bash out

$$\text{div}\mathbf{a} = \frac{1}{r^2}\frac{\partial(r^2 a_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(a_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi}$$

In glorious detail this is

$$\begin{aligned} \text{div}\mathbf{a} &= 3\left(\sin\theta\cos\phi c_x + \sin\theta\sin\phi c_y + \cos\theta c_z\right) + \\ &\qquad \frac{1}{\sin\theta}\left((\cos^2\theta - \sin^2\theta)(\cos\phi c_x + \sin\phi c_y) - 2\sin\theta\cos\theta c_z\right) + \\ &\qquad \frac{1}{\sin\theta}\left(-\cos\phi c_x - \sin\phi c_y\right) \end{aligned}$$

A bit more bashing and you'll find

$$\begin{aligned} \text{div}\mathbf{a} &= \sin\theta\cos\phi c_x + \sin\theta\sin\phi c_y + \cos\theta c_z \\ &= \hat{\mathbf{e}}_r \cdot \mathbf{c} \end{aligned}$$

This is EXACTLY what you worked out before of course.

Take home messages from these examples:

- Just as physical vectors are independent of their coordinate systems, so are differential operators.
- Don't forget about the vector geometry you did in the 1st year. Rotation matrices are useful!
- Spherical polars were NOT a good coordinate system in which to think about this problem. Let the symmetry guide you.

Lecture 7

Gauss' and Stokes' Theorems

This section finally begins to deliver on why we introduced div grad and curl. Two theorems, both of them over two hundred years old, are explained:

- **Gauss' Theorem** enables an integral taken over a volume to be replaced by one taken over the surface bounding that volume, and vice versa. Why would we want to do that? Computational efficiency and/or numerical accuracy!

- **Stokes' Law** enables an integral taken around a closed curve to be replaced by one taken over *any* surface bounded by that curve.

7.1 Gauss' Theorem

Suppose that $\mathbf{a}(\mathbf{r})$ is a vector field and we want to compute the total flux of the field across the surface S that bounds a volume V. That is, we are interested in calculating:

$$\int_S \mathbf{a} \cdot d\mathbf{S}$$

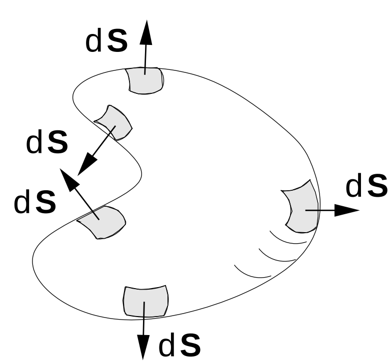


Figure 7.1: The surface element $d\mathbf{S}$ must stick out of the surface.

where recall that $d\mathbf{S}$ is normal to the locally planar surface element and **must everywhere point out of the volume** as shown in Figure 7.1.

Gauss' Theorem tells us that we can do this by considering the total flux generated inside the volume V:

Gauss' Theorem

$$\int_S \mathbf{a} \cdot d\mathbf{S} = \int_V \text{div } \mathbf{a} \, dV$$

obtained by integrating the divergence over the entire volume.

7.2 Informal proof

An non-rigorous proof can be realized by recalling that we defined div by considering the efflux dE from the surfaces of an infinitesimal volume element

$$dE = \mathbf{a} \cdot d\mathbf{S}$$

and defining it as

$$\text{div } \mathbf{a} \, dV \quad = \quad dE \quad = \quad \mathbf{a} \cdot d\mathbf{S} \; .$$

If we sum over the volume elements, this results in a sum over the surface elements. But if two elemental surface touch, their $d\mathbf{S}$ vectors are in opposing direction and cancel as shown in Figure 7.2. Thus the sum over surface elements gives the overall bounding surface.

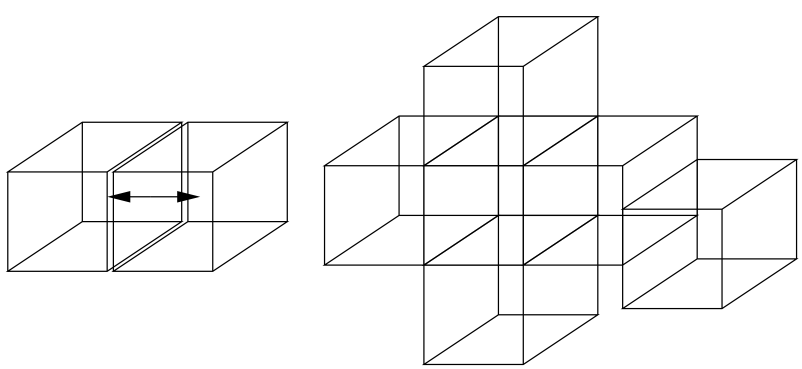


Figure 7.2: When two elements touch, the $d\mathbf{S}$ vectors at the common surface cancel out. One can imagine building the entire volume up from the infinitesimal units.

♣ Example of Gauss' Theorem

This is a typical example, in which the surface integral is rather tedious, whereas the volume integral is straightforward.

Q Derive $\int_S \mathbf{a} \cdot d\mathbf{S}$ where $\mathbf{a} = z^3\hat{\boldsymbol{k}}$ and S is the surface of a sphere of radius R centred on the origin:

1. directly;
2. by applying Gauss' Theorem

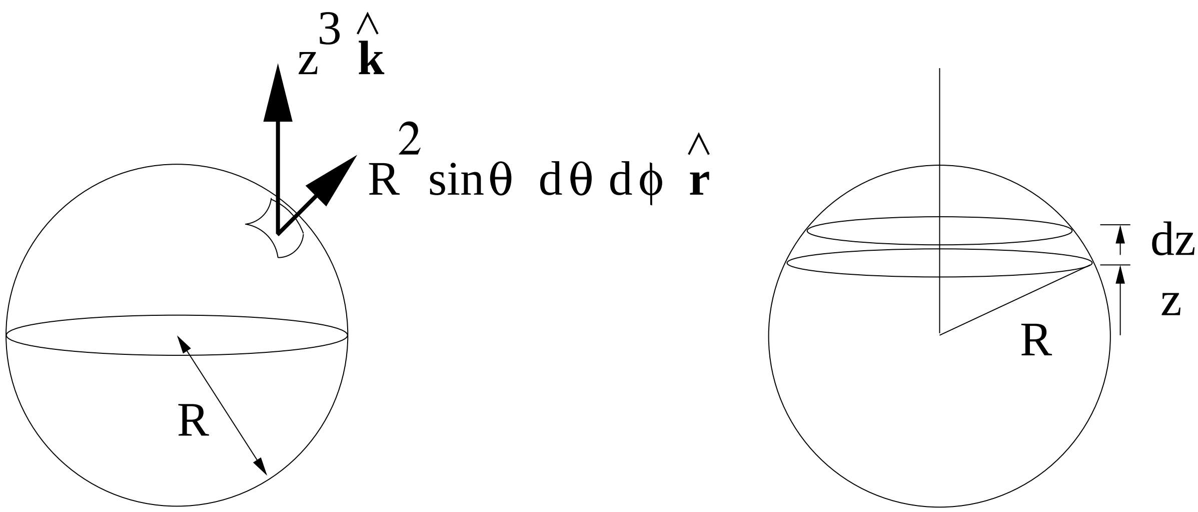


Figure 7.3:

A (1) On the surface of the sphere, $\mathbf{a} = R^3\cos^3\theta\hat{\boldsymbol{k}}$ and $d\mathbf{S} = R^2\sin\theta d\theta d\phi\hat{\mathbf{r}}$. Everywhere $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{k}} = \cos\theta$.

$$\begin{aligned}
\Rightarrow \int_S \mathbf{a} \cdot d\mathbf{S} &= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} R^3\cos^3\theta\ .\ R^2\sin\theta d\theta d\phi\hat{\mathbf{e}}_r \cdot \hat{\boldsymbol{k}} \\
&= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} R^3\cos^3\theta\ .\ R^2\sin\theta d\theta d\phi\ .\ \cos\theta \\
&= 2\pi R^5 \int_0^{\pi} \cos^4\theta\sin\theta d\theta \\
&= \frac{2\pi R^5}{5}\left[-\cos^5\theta\right]_0^{\pi} = \frac{4\pi R^5}{5}
\end{aligned}$$

(2) To apply Gauss' Theorem, we need to figure out div $\mathbf{a}$ and decide how to compute the volume integral. The first is easy:

$$\text{div}\mathbf{a} = 3z^2$$

For the second, because div$\mathbf{a}$ involves just z, we can divide the sphere into discs of constant z and thickness dz, as shown in Fig. 7.3. Then

$$dV = \pi(R^2 - z^2)dz$$

and

$$\begin{aligned}\int_V \text{div } \mathbf{a} dV &= 3\pi \int_{-R}^{R} z^2(R^2 - z^2)dz \\ &= 3\pi \left[\frac{R^2 z^3}{3} - \frac{z^5}{5}\right]_{-R}^{R} \\ &= \frac{4\pi R^5}{5}\end{aligned}$$

7.3 Surface versus volume integrals

At first sight, it might seem that with a computer performing surface integrals might be better than a volume integral, perhaps because there are, somehow, "fewer elements". However, this is not the case. Imagine doing a surface integral over a wrinkly surface, say that of the moon. All the elements involved in the integration are "difficult" and must be modelled correctly. With a volume integral, most of the elements are not at the surface, and so the bulk of the integral is done without accurate modelling. The computation easier, faster, and better conditioned numerically.

7.4 Extension to Gauss' Theorem

Suppose the vector field $\mathbf{a}(\mathbf{r})$ is of the form $\mathbf{a} = U(\mathbf{r})\mathbf{c}$, where $U(\mathbf{r})$ as scalar field and $\mathbf{c}$ is a constant vector. Then, as we showed in the previous lecture,

$$\begin{aligned}\text{div } \mathbf{a} &= \text{grad} U \cdot \mathbf{c} + U \text{div } \mathbf{c} \\ &= \text{grad} U \cdot \mathbf{c}\end{aligned}$$

since div$\mathbf{c} = 0$ because $\mathbf{c}$ is constant.

Gauss' Theorem becomes

$$\int_S U\mathbf{c} \cdot d\mathbf{S} = \int_V \text{grad } U \cdot \mathbf{c} dV$$

or, alternatively, taking the constant $\mathbf{c}$ out of the integrals

$$\mathbf{c} \cdot \left(\int_S U d\mathbf{S}\right) = \mathbf{c} \cdot \left(\int_V \text{grad } U dV\right)$$

This is still a scalar equation but we now note that the vector **c** is arbitrary so that the result must be true for any vector **c**. This can be true only if the vector equation

$$\int_S U d\mathbf{S} = \int_V \text{grad } U dV$$

is satisfied.

If you think this is fishy, just write $\mathbf{c} = \hat{\boldsymbol{\imath}}$, then $\mathbf{c} = \hat{\boldsymbol{\jmath}}$, and $\mathbf{c} = \hat{\boldsymbol{k}}$ in turn, and you must obtain the three components of $\int_S U d\mathbf{S}$ in turn.

Further "extensions" can be obtained of course. For example one might be able to write the vector field of interest as

$$\mathbf{a}(\mathbf{r}) = \mathbf{b}(\mathbf{r}) \times \mathbf{c}$$

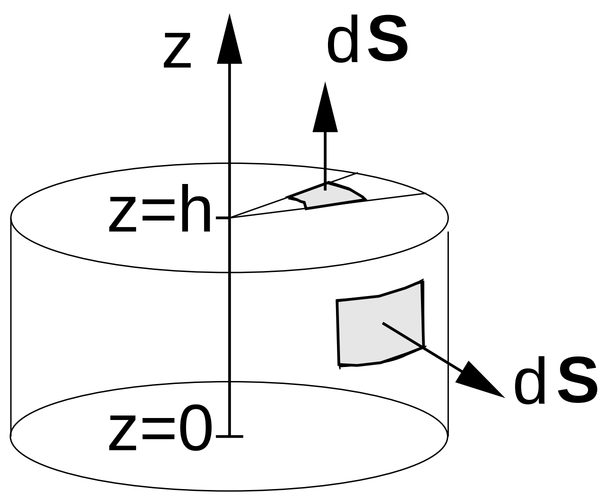
where **c** is a constant vector.

♣ Example of extension to Gauss' Theorem

Q $U = x^2 + y^2 + z^2$ is a scalar field, and volume V is the cylinder $x^2 + y^2 \leq a^2$, $0 \leq z \leq h$. Compute the surface integral

$$\int_S U d\mathbf{S}$$

over the surface of the cylinder.



A It is immediately clear from symmetry that there is no contribution from the curved surface of the cylinder since for every vector surface element there exists an equal and opposite element with the same value of U. We therefore need consider only the top and bottom faces.

Top face:

$$U = x^2 + y^2 + z^2 = r^2 + h^2 \text{ and } d\mathbf{S} = r dr d\phi \hat{\boldsymbol{k}}$$

so

$$\int U d\mathbf{S} = \int_{r=0}^{a} (h^2 + r^2) 2\pi r dr \int_{\phi=0}^{2\pi} d\phi \hat{\boldsymbol{k}} = \hat{\boldsymbol{k}} \pi \left[h^2 r^2 + \frac{1}{2} r^4 \right]_0^a = \pi [h^2 a^2 + \frac{1}{2} a^4] \hat{\boldsymbol{k}}$$

Bottom face:

$$U = r^2 \text{ and } d\mathbf{S} = -r\,dr\,d\phi\hat{\boldsymbol{k}}$$

The contribution from this face is thus $-\frac{\pi a^4}{2}\hat{\boldsymbol{k}}$, and the total integral is $\pi h^2 a^2 \hat{\boldsymbol{k}}$.

On the other hand, using Gauss' Theorem we have to compute

$$\int_V \text{grad } U dV$$

In this case, grad $U = 2\mathbf{r}$,

$$2\int_V (x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{k}} + z\hat{\boldsymbol{k}})r\ dr\ dz\ d\phi$$

The integrations over x and y are zero by symmetry, so that the only remaining part is

$$2\int_{z=0}^{h} zdz \int_{r=0}^{a} r\ dr \int_{\phi=0}^{2\pi} d\phi\hat{\boldsymbol{k}} = \pi a^2 h^2 \hat{\boldsymbol{k}}$$

7.5 Stokes' Theorem

Stokes' Theorem relates a line integral around a closed path to a surface integral over what is called a *capping surface* of the path.

Stokes' Theorem states:

$$\oint_C \mathbf{a} \cdot d\mathbf{l} = \int_S \text{curl }\ \mathbf{a} \cdot d\mathbf{S}$$

where S is *any* surface capping the curve C.

Why have we used $d\mathbf{l}$ rather than $d\mathbf{r}$, where $\mathbf{r}$ is the position vector?

There is no good reason for this, as $d\mathbf{l} = d\mathbf{r}$. It just seems to be common usage in line integrals!

7.6 Informal proof

You will recall that in Lecture 5 that we defined curl as the circulation per unit area, and showed that

$$\sum_{\text{around elemental loop}} \mathbf{a}\cdot d\mathbf{l} = dC = (\boldsymbol{\nabla}\times\mathbf{a})\cdot d\mathbf{S} \quad .$$

Now if we add these little loops together, the internal line sections cancel out because the $d\mathbf{l}$'s are in opposite direction but the field $\mathbf{a}$ is not. This gives the larger surface and the larger bounding contour as shown in Fig. 7.4.

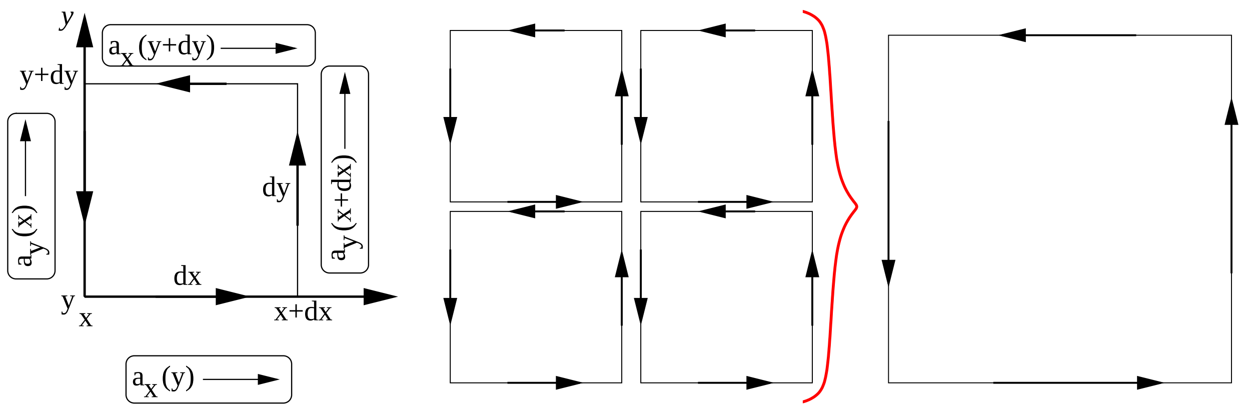


Figure 7.4: An example of an elementary loop, and how they combine together.

For a given contour, the capping surface can be ANY surface bound by the contour. The only requirement is that the surface element vectors point in the "general direction" of a right-handed screw with respect to the sense of the contour integral. See Fig. 7.5.

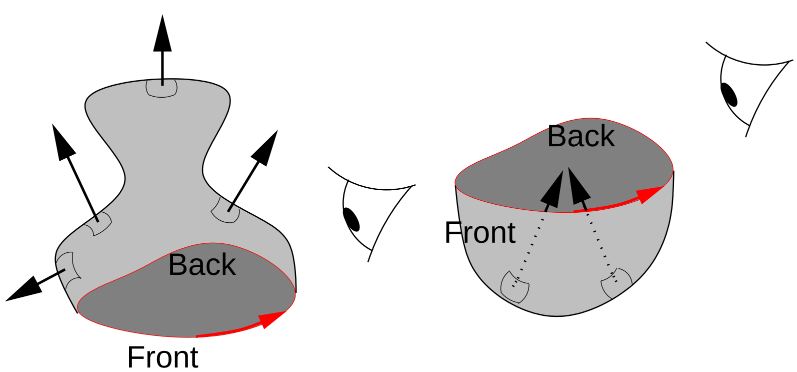


Figure 7.5: For a given contour, the bounding surface can be any shape. $d\mathbf{S}$'s must have a positive component in the sense of a r-h screw wrt the contour sense.

♣ Example of Stokes' Theorem

In practice, (and especially in exam questions!) the bounding contour is often planar, and the capping surface flat or hemispherical or cylindrical.

Q Vector field $\mathbf{a} = x^3\hat{\boldsymbol{\jmath}} - y^3\hat{\boldsymbol{\imath}}$ and C is the circle of radius R centred on the origin. Derive

$$\oint_C \mathbf{a} \cdot d\mathbf{l}$$

directly and (ii) using Stokes' theorem where the surface is the planar surface bounded by the contour.

A(i) Directly. On the circle of radius R

$$\mathbf{a} = R^3(-\sin^3\theta\hat{\boldsymbol{\imath}} + \cos^3\theta\hat{\boldsymbol{\jmath}})$$

and

$$d\mathbf{l} = Rd\theta(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}})$$

so that:

$$\oint_C \mathbf{a} \cdot d\mathbf{l} = \int_0^{2\pi} R^4(\sin^4\theta + \cos^4\theta)d\theta = \frac{3\pi}{2}R^4,$$

since

$$\int_0^{2\pi} \sin^4\theta d\theta = \int_0^{2\pi} \cos^4\theta d\theta = \frac{3\pi}{4}$$

A(ii) Using Stokes' theorem ...

$$\text{curl } \mathbf{a} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = 3(x^2 + y^2)\hat{\mathbf{k}} = 3r^2\hat{\mathbf{k}}$$

We choose area elements to be circular strips of radius r thickness dr. Then

$$d\mathbf{S} = 2\pi r dr\hat{\mathbf{k}} \qquad \text{and} \qquad \int_S \text{curl } \mathbf{a} \cdot d\mathbf{S} = 6\pi\int_0^R r^3 dr = \frac{3\pi}{2}R^4$$

7.7 An Extension to Stokes' Theorem

Just as we considered one extension to Gauss' theorem (not *really* an extension, more of a re-expression), so we will try something similar with Stoke's Theorem.

Again let $\mathbf{a}(\mathbf{r}) = U(\mathbf{r})\mathbf{c}$, where $\mathbf{c}$ is a constant vector. Then

$$\text{curl } \mathbf{a} = U\text{curl } \mathbf{c} + \text{grad } U \times \mathbf{c})$$

Again, curl $\mathbf{c}$ is zero. Stokes' Theorem becomes in this case:

$$\oint_C U(\mathbf{c} \cdot d\mathbf{l}) = \int_S (\text{grad } U \times \mathbf{c} \cdot d\mathbf{S} = \int_S \mathbf{c} \cdot (d\mathbf{S} \times \text{grad } U)$$

or, rearranging the triple scalar products and taking the constant $\mathbf{c}$ out of the integrals gives

$$\mathbf{c} \cdot \oint_C U d\mathbf{l} = -\mathbf{c} \cdot \int_S \text{grad} U \times d\mathbf{S} \quad .$$

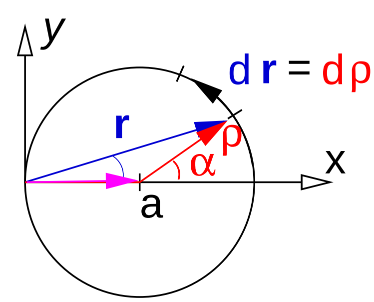
But $\mathbf{c}$ is arbitrary and so

$$\oint_C U d\mathbf{l} = -\int_S \text{grad } U \times d\mathbf{S}$$

7.8 ♣ Example of extension to Stokes' Theorem

Q Derive $\oint_C U d\mathbf{r}$ (i) directly and (ii) using Stokes', where $U = x^2 + y^2 + z^2$ and the line integral is taken around C the circle $(x-a)^2 + y^2 = a^2$ and $z = 0$.

Note that, for no special reason, we have used $d\mathbf{r}$ here not $d\mathbf{l}$.



A(i) First some preamble.

If the circle were centred at the origin, we would write $d\mathbf{r} = ad\theta\hat{\mathbf{e}}_\theta = ad\theta(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}})$. For such a circle the magnitude $r = |\mathbf{r}| = a$, a constant and so $dr = 0$.

However, in this example $d\mathbf{r}$ is not always in the direction of $\hat{\mathbf{e}}_\theta$, and $dr \neq 0$. Could you write down $d\mathbf{r}$? If not, revise Lecture 3, where we saw that in plane polars $x = r\cos\theta$, $y = r\sin\theta$ and the general expression is

$$d\mathbf{r} = dx\hat{\boldsymbol{\imath}} + dy\hat{\boldsymbol{\jmath}} = (\cos\theta dr - r\sin\theta d\theta)\hat{\boldsymbol{\imath}} + (\sin\theta dr + r\cos\theta d\theta)\hat{\boldsymbol{\jmath}}$$

To avoid having to find an expression for r in terms of θ, we will perform a coordinate transformation by writing $\mathbf{r} = [a, 0]^\top + \boldsymbol{\rho}$. So, $x = (a + \rho\cos\alpha)$ and $y = \rho\sin\alpha$, and on the circle itself where $\rho = a$

$$\mathbf{r} = a(1+\cos\alpha)\hat{\boldsymbol{\imath}} + a\sin\alpha\hat{\boldsymbol{\jmath}} \quad ,$$

$$d\mathbf{r} = ad\alpha(-\sin\alpha\hat{\boldsymbol{\imath}} \; + \; \cos\alpha\hat{\boldsymbol{\jmath}}) \quad ,$$

and, as $z = 0$ on the circle,

$$U = a^2(1+\cos\alpha)^2 + a^2\sin^2\alpha = 2a^2(1+\cos\alpha) \; .$$

The line integral becomes

$$\oint U d\mathbf{r} = 2a^3 \int_{\alpha=0}^{2\pi} (1+\cos\alpha)(-\sin\alpha\hat{\boldsymbol{\imath}} + \cos\alpha\hat{\boldsymbol{\jmath}})d\alpha = 2\pi a^3\hat{\boldsymbol{\jmath}}$$

A(ii) Now using Stokes' ...

For a planar surface covering the disc, the surface element can be written using the new parametrization as

$$d\mathbf{S} = \rho \; d\rho \; d\alpha\hat{\boldsymbol{k}}$$

Remember that $U = x^2 + y^2 + z^2 = r^2$, and as $z = 0$ in the plane

$$\text{grad } U = 2(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}) = 2(a+\rho\cos\alpha)\hat{\boldsymbol{\imath}} + 2\rho\sin\alpha\hat{\boldsymbol{\jmath}} \; .$$

Be careful to note that x, y are specified for any point on the disc, not on its circular boundary!

So

$$d\mathbf{S}\times\text{grad}U = 2\rho \; d\rho \; d\alpha \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 0 & 0 & 1 \\ (a+\rho\cos\alpha) & \rho\sin\alpha & 0 \end{vmatrix} = 2\rho[-\rho\sin\alpha\hat{\boldsymbol{\imath}} + (a+\rho\cos\alpha)\hat{\boldsymbol{\jmath}}] \; d\rho \; d\alpha$$

Both $\int_0^{2\pi} \sin\alpha d\alpha = 0$ and $\int_0^{2\pi} \cos\alpha d\alpha = 0$, so we are left with

$$\int_S d\mathbf{S} \times \text{grad}U = \int_{\rho=0}^{a} \int_{\alpha=0}^{2\pi} 2\rho a\hat{\boldsymbol{\jmath}} \; d\rho \; d\alpha = 2\pi a^3\hat{\boldsymbol{\jmath}}$$

Lecture 8

Engineering applications

In Lecture 6 we saw one classic example of the application of vector calculus to Maxwell's equation.

In this lecture we explore a few more examples from fluid mechanics and heat transfer. As with Maxwell's eqations, the examples show how vector calculus provides a powerful way of representing underlying physics.

The power come from the fact that div, grad and curl have a significance or meaning which is more immediate than a collection of partial derivatives. Vector calculus will, with practice, become a convenient shorthand for you.

- Electricity – Ampère's Law
- Fluid Mechanics - The Continuity Equation
- Thermo: The Heat Conduction Equation
- Mechanics/Electrostatics - Conservative fields
- The Inverse Square Law of force
- Gravitational field due to distributed mass
- Gravitational field inside body
- Pressure forces in non-uniform flows

8.1 Electricity – Ampère's Law

If the frequency is low, the displacement current in Maxwell's equation curl$\mathbf{H} = \mathbf{J} + \partial\mathbf{D}/\partial t$ is negligible, and we find

$$\text{curl}\mathbf{H} = \mathbf{J}$$

Hence

$$\int_S \text{curl}\mathbf{H} \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

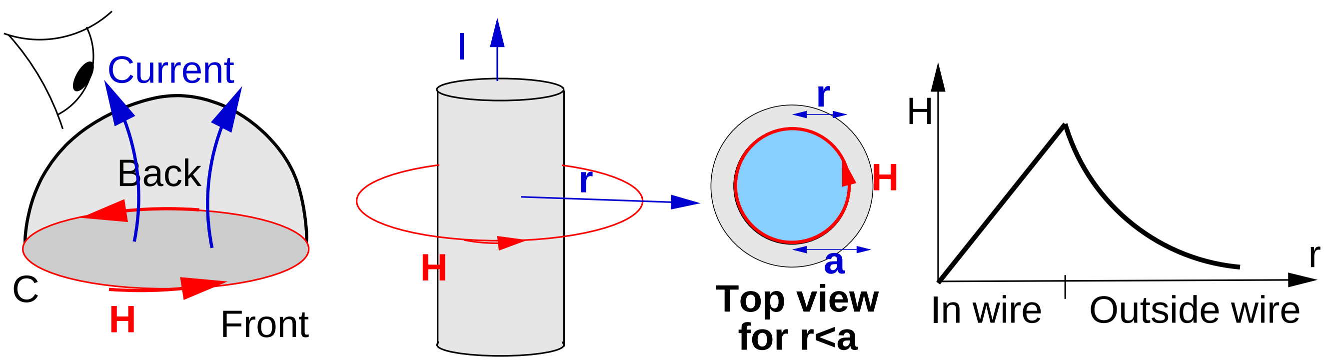
or

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

where $\int_S \mathbf{J} \cdot d\mathbf{S}$ is total current through the surface.

Now consider the $\mathbf{H}$ around a straight wire carrying current I. Symmetry tells us the $\mathbf{H}$ is in the $\hat{\mathbf{e}}_\theta$ direction, in a rhs screw sense with respect to the current. (You might check this against Biot-Savart's law.)

Suppose we asked what is the magnitude of $\mathbf{H}$?



Inside the wire, the bounding contour only encloses a fraction $(\pi r^2)/(\pi a^2)$ of the current, and so

$$\begin{aligned} H2\pi r &= \int \mathbf{J} \cdot d\mathbf{S} = I(r^2/A^2) \\ \Rightarrow\ H &= Ir/2\pi A^2 \end{aligned}$$

whereas outside we enclose all the current, and so

$$\begin{aligned} H2\pi r &= \int \mathbf{J} \cdot d\mathbf{S} = I \\ \Rightarrow\ H &= I/2\pi r \end{aligned}$$

A plot is shown in the Figure.

8.2 Fluid Mechanics - The Continuity Equation

The **Continuity Equation** expresses the condition of conservation of mass in a fluid flow. The continuity principle applied to *any* volume (called a *control volume*) may be expressed in words as follows:

> "The net rate of mass flow of fluid out of the control volume must equal the rate of decrease of the mass of fluid within the control volume"

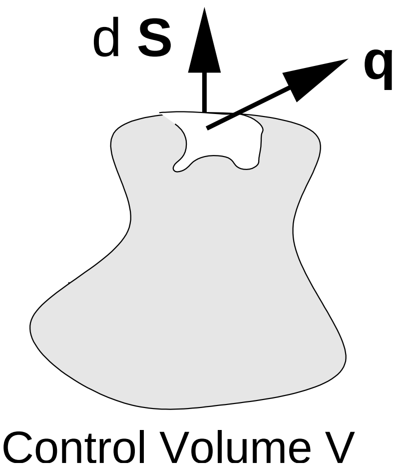


Figure 8.1:

To express the above as a mathematical equation, we denote the velocity of the fluid at each point of the flow by $\mathbf{q}(\mathbf{r})$ (a vector field) and the density by $\rho(\mathbf{r})$ (a scalar field). The element of rate-of-volume-loss through surface $d\mathbf{S}$ is $d\dot{V} = \mathbf{q}\cdot d\mathbf{S}$, so the rate of mass loss is

$$d\dot{M} = \rho\mathbf{q} \cdot d\mathbf{S},$$

so that the total rate of mass loss from the volume is

$$-\frac{\partial}{\partial t}\int_V \rho(\mathbf{r})dV = \int_S \rho\mathbf{q} \cdot d\mathbf{S}.$$

Assuming that the volume of interest is fixed, this is the same as

$$-\int_V \frac{\partial\rho}{\partial t}dV = \int_S \rho\mathbf{q} \cdot d\mathbf{S}\ .$$

Now we use Gauss' Theorem to transform the RHS into a volume integral

$$-\int_V \frac{\partial\rho}{\partial t}dV = \int_V \text{div}\ (\rho\mathbf{q})dV\ .$$

The two volume integrals can be equal for any control volume V only if the two integrands are equal at each point of the flow. This leads to the mathematical formulation of

The Continuity Equation:

$$\text{div }(\rho\mathbf{q}) = -\frac{\partial\rho}{\partial t}$$

Notice that if the density doesn't vary with time, div $(\rho\mathbf{q}) = 0$, and if the density doesn't vary with position then

The Continuity Equation for uniform, time-invariant density:

$$\text{div }(\mathbf{q}) = 0 \quad .$$

In this last case, we can say that the flow **q** is solenoidal.

8.3 Thermodynamics - The Heat Conduction Equation

Flow of heat is very similar to flow of fluid, and heat flow satisfies a similar continuity equation. The flow is characterized by the heat current density $\mathbf{q}(\mathbf{r})$ (heat flow per unit area and time), sometimes misleadingly called heat flux.

Assuming that there is no mass flow across the boundary of the control volume and no source of heat inside it, the rate of flow of heat out of the control volume by conduction must equal the rate of decrease of internal energy (constant volume) or enthalpy (constant pressure) within it. This leads to the equation

$$\text{div }\ \mathbf{q} = -\rho c\frac{\partial T}{\partial t},$$

where ρ is the density of the conducting medium, c its specific heat (both are assumed constant) and T is the temperature.

In order to solve for the temperature field another equation is required, linking q to the temperature gradient. This is

$$\mathbf{q} = -\kappa\text{grad }\ T,$$

where κ is the thermal conductivity of the medium. Combining the two equations gives the *heat conduction equation*:

$$-\text{div }\mathbf{q} = \kappa\text{div }\ \text{grad }\ T = \kappa\nabla^2 T = \rho c\frac{\partial T}{\partial t}$$

where it has been assumed that κ is a constant. In steady flow the temperature field satisfies Laplace's Equation $\nabla^2 T = 0$.

8.4 Mechanics - Conservative fields of force

A conservative field of force is one for which the work done

$$\int_A^B \mathbf{F} \cdot d\mathbf{r},$$

moving from A to B is indep. of path taken. As we saw in Lecture 4, conservative fields must satisfy the condition

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0,$$

Stokes' tells us that this is

$$\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0,$$

where S is *any* surface bounded by C.

But if true for *any* C containing A and B, it must be that

$$\text{curl } \mathbf{F} = \mathbf{0}$$

Conservative fields are irrotational
All radial fields are irrotational

One way (actually the only way) of satisfying this condition is for

$$\mathbf{F} = \boldsymbol{\nabla}\, U$$

The scalar field $U(\mathbf{r})$ is the Potential Function

8.5 The Inverse Square Law of force

Radial forces are found in electrostatics and gravitation — so they are certainly irrotational and conservative.

But in nature these radial forces are also inverse square laws. One reason why this may be so is that it turns out to be the only central force field which is **solenoidal**, i.e. has zero divergence.

If $\mathbf{F} = f(r)\mathbf{r}$,

$$\text{div } \mathbf{F} = 3f(r) + rf'(r).$$

For div $\mathbf{F} = 0$ we conclude

$$r\frac{df}{dr} + 3f = 0$$

or

$$\frac{df}{f} + 3\frac{dr}{r} = 0.$$

Integrating with respect to r gives $fr^3 = const = A$, so that

$$\mathbf{F} = \frac{A\mathbf{r}}{r^3}, \quad |\mathbf{F}| = \frac{A}{r^2}.$$

The condition of zero divergence of the inverse square force field applies everywhere except at $\mathbf{r} = \mathbf{0}$, where the divergence is infinite.

To show this, calculate the outward normal flux out of a sphere of radius R centered on the origin when $\mathbf{F} = F\hat{\mathbf{r}}$. This is

$$\int_S \mathbf{F} \cdot d\mathbf{S} = F \int_{\text{Sphere}} \hat{\mathbf{r}} \cdot d\mathbf{S} = F \int_{\text{Sphere}} d = F4\pi R^2 = 4\pi A = \text{Constant}.$$

Gauss tells us that this flux must be equal to

$$\int_V \text{div } \mathbf{F} dV = \int_0^R \text{div } \mathbf{F} 4\pi r^2 dr$$

where we have done the volume integral as a summation over thin shells of surface area $4\pi r^2$ and thickness dr.

But for all finite r, div$\mathbf{F} = 0$, so div$\mathbf{F}$ must be infinite at the origin.

The flux integral is thus

- zero — for any volume which does not contain the origin
- $4\pi A$ for any volume which does contain it.

8.6 Gravitational field due to distributed mass: Poisson's Equation

If one tried the same approach as §8.4 for the gravitational field, $A = Gm$, where m is the mass at the origin and G the universal gravitational constant, one would run into the problem that there is no such thing as point mass.

We can make progress though by considering distributed mass.

The mass contained in each small volume element dV is ρdV and this will make a contribution $-4\pi\rho G dV$ to the flux integral from the control volume. Mass outside the control volume makes no contribution, so that we obtain the equation

$$\int_S \mathbf{F} \cdot d\mathbf{S} = -4\pi G \int_V \rho dV.$$

Transforming the left hand integral by Gauss' Theorem gives

$$\int_V \text{div } \mathbf{F} dV = -4\pi G \int_V \rho dV$$

which, since it is true for any V, implies that

$$-\text{div } \mathbf{F} = 4\pi\rho G.$$

Since the gravitational field is also conservative (i.e. irrotational) it must have an associated potential function U, so that $\mathbf{F} = \text{grad } U$. It follows that the gravitational potential U satisfies

Poisson's Equation

$$\nabla^2 U = 4\pi\rho G \ .$$

Using the integral form of Poisson's equation, it is possible to calculate the gravitational field inside a spherical body whose density is a function of radius only. We have

$$4\pi R^2 F = 4\pi G \int_0^R 4\pi r^2 \rho dr,$$

where $F = |\mathbf{F}|$, or

$$|F| = \frac{G}{R^2} \int_0^R 4\pi r^2 \rho dr = \frac{MG}{R^2},$$

where M is the total mass inside radius R. For the case of uniform density, this is equal to $M = \frac{4}{3}\pi\rho R^3$ and $|F| = \frac{4}{3}\pi\rho G R$.

8.7 Pressure forces in non-uniform flows

When a body is immersed in a flow it experiences a net pressure force

$$\mathbf{F}_p = -\int_S p d\mathbf{S},$$

where S is the surface of the body. If the pressure p is non-uniform, this integral is not zero. The integral can be transformed using Gauss' Theorem to give the alternative expression

$$\mathbf{F}_p = -\int_V \text{grad } p \, dV,$$

where V is the volume of the body. In the simple hydrostatic case $p + \rho g z = \mathit{constant}$, so that

$$\text{grad } p = -\rho g k$$

and the net pressure force is simply

$$\mathbf{F}_p = g\hat{\mathbf{k}} \int_V \rho dV$$

which, in agreement with Archimedes' principle, is equal to the weight of fluid displaced.