2A1D Vector Algebra and Calculus II

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Course page www.robots.ox.ac.uk/∼ian/Teaching/Vectors
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1. (a) Show how the definition of the gradient of a scalar function $U(x, y, z)$

$$\text{grad } U = \hat{\mathbf{i}}\frac{\partial V}{\partial x} + \hat{\mathbf{j}}\frac{\partial V}{\partial y} + \hat{\mathbf{k}}\frac{\partial V}{\partial z}$$

is equivalent to the following:

(i) the component of grad U in any direction is the rate of change of U with respect to distance in that direction;

(ii) grad U is a vector whose magnitude at any point is equal to the greatest rate of change of U with respect to distance at that point and whose direction is that of the greatest rate of change.

(b) Show that grad U is perpendicular to the surface U = constant.

(c) Derive the gradients of the following functions:

(i) x^2y; (ii) $\log r$; (iii) $(\mathbf{c} \cdot \mathbf{r})^2$, for $\mathbf{c}$ = constant vector, where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

(d) What is a directional derivative? Compute the directional derivative for $U = x^2y$ at the point $[1, 1, 1]$ in the direction $[1, 2, 0]/\sqrt{5}$.

(e) Verify your result by evaluating values of U at position $\mathbf{r} = [1, 1, 1]$ then at $\mathbf{r} = [1 + \delta, 1 + 2\delta, 1]$, where δ is any small number. (For example $\delta = 0.01$.) Find the change in U, and divide it by the *distance* moved between the two positions.

2. (a) Using Cartesian co-ordinates, show that div$\mathbf{a}$dV is the outward normal flux of the vector $\mathbf{a}$ from the volume element dV.

(b) Derive the divergences of the following vector fields:

(i) $\mathbf{r}$; (ii) $r^n\mathbf{r}$; (iii) $r^n\mathbf{c}$

where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, $r = |\mathbf{r}|$ and $\mathbf{c}$ is a constant vector.

3. (a) Using Cartesian co-ordinates, show that $(\text{curl } \mathbf{a})_z \mathrm{d}S$ is the circulation of the vector $\mathbf{a}$ around the perimeter of the area element $\mathrm{d}S$ in the x, y plane.

   (b) Write down the curl of each of the following vector fields:
   (i) $x^2 y \hat{\mathbf{k}}$; (ii) $r^n \mathbf{r}$; (iii) $\mathbf{r} \times \mathbf{c}$, where $\mathbf{c}$ is a constant vector.

   (c) Show that operators "curl grad" and "div curl" are identically zero. (NB, the scalar and vector fields used must the general $f(\mathbf{r})$ and $\mathbf{f}(\mathbf{r})$, and not specific examples.)

4. (a) Show that if (u, v, w) is a set of curvilinear co-ordinates, the elements of length corresponding to small changes $\mathrm{d}u, \mathrm{d}v$ and $\mathrm{d}w$ are $h_1 \mathrm{d}u$, $h_2 \mathrm{d}v$ and $h_3 \mathrm{d}w$ respectively, where

$$h_1 = \left| \frac{\partial \mathbf{r}}{\partial u} \right|$$

   and similarly for $h_{2,3}$.

   (b) Derive expressions for the h's for the following co-ordinate systems:

      i. cylindrical polars
      ii. spherical polars
      iii. $x = uv \cos w, \qquad y = uv \sin w, \qquad z = (u^2 - v^2)/2.$

   (c) Hence obtain expessions for $\nabla^2 U$ in all three co-ordinate systems. (The general formula in the lecture notes may be assumed without proof but you should understand the principles on which it is based.)

5. (a) State the divergence theorem of Gauss.

   (b) Show using surface integration that if $\mathbf{a} = (x^3, y^3, z^3)$ then

$$\int_S \mathbf{a} \cdot \mathrm{d}\mathbf{S} = \frac{12}{5} \pi R^5$$

   where the integration is over the sphere $x^2 + y^2 + z^2 = R^2$.
   (You may assume $\int_0^{2\pi} (\cos^4 \phi + \sin^4 \phi) \mathrm{d}\phi = 3\pi/2$.)

   (c) Verify your result by evaluating

$$\int_V \text{div } \mathbf{a} \;\; \mathrm{d}V$$

   throughout the volume of the sphere.

6. (a) State Stokes' theorem, explaining carefully how the surface orientation and direction of line integral are related.

   (b) Evaluate the line integral $\oint \mathbf{F} \cdot d\mathbf{r}$ around the circumference of the circle $x^2 + y^2 = a^2$, $z = 0$, where $\mathbf{F}$ is the vector $[0, x^3, 0]$.

   (c) Verify your result by evaluating $\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$

      i. where S is the flat surface enclosed by the circle $x^2 + y^2 = a^2$, $z = 0$.

      ii. where S is the hemispherical surface $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

7. (a) Show that $\text{div } (U\mathbf{a}) = \mathbf{a} \cdot \text{grad } U + U\text{div } \mathbf{a}$.

   (b) Using Gauss' theorem and the result of part (a) show that

$$\int_V \text{grad } U dV = \int_S U d\mathbf{S}$$

   where the volume V is enclosed by the surface S.

   (c) Verify this result for the case where $U = z^3$, S is the surface $x^2 + y^2 + z^2 = a^2$, and V is its interior.

8. (a) Prove the identity $\text{div } (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \text{curl}\mathbf{u} - \mathbf{u} \cdot \text{curl } \mathbf{v}$

   (b) Hence, by means of the divergence (Gauss') theorem, show that

$$\int_S (\text{grad } \psi \times \mathbf{v}) \cdot d\mathbf{S} = -\int_V (\text{grad } \psi \cdot \text{curl } \mathbf{v}) dV$$

   where the first integral is over the surface S enclosing the volume V of a simple body and the second integral is over this volume.

   (c) Verify the result of part (b) for a right circular cylinder of radius a and height h, resting on the xy plane, with its axis coincident with the z axis when

$$\psi = z^2 \quad \text{and} \quad \mathbf{v} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}} \;.$$

9. Integral equations of continuity and momentum for an inviscid fluid may be derived in the form

$$\int_V \frac{\partial \rho}{\partial t} \mathrm{d}V + \int_S \rho \mathbf{q} \cdot \mathrm{d}\mathbf{S} = 0 \qquad \text{and} \qquad F_x = \int_V \frac{\partial}{\partial t}(\rho u) \mathrm{d}V + \int_S \rho u \mathbf{q} \cdot \mathrm{d}\mathbf{S}$$

where S is a closed surface containing the volume V, ρ is the fluid density, $\mathbf{q}$ is the fluid velocity vector and u its component in the x-direction, and F_x is the force on the control volume in the x-direction.

(a) By means of the divergence theorem, derive the equations

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\,(\rho \mathbf{q}) = 0 \qquad \text{and} \qquad F_x = \int_V \rho \left(\frac{\partial u}{\partial t} + \mathbf{q} \cdot \nabla u \right) \mathrm{d}V.$$

(b) Hence deduce that the acceleration of a fluid particle in the x-direction is

$$\ddot{x} = \frac{\partial u}{\partial t} + \mathbf{q} \cdot \nabla u \ .$$

(c) The fluid particle velocity is $\mathbf{q} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}$. Derive an expression for the 3D fluid particle acceleration vector in terms of $\mathbf{q}$ and the operator $(\mathbf{q} \cdot \nabla)$.

Answers

1. (c) (i) $2xy\hat{\mathbf{i}} + x^2\hat{\mathbf{j}}$ (ii) $\mathbf{r}/r^2$ (iii) $2(\mathbf{c} \cdot \mathbf{r})\mathbf{c}$ (d) Directional derivative is $4/\sqrt{5}$.
2. (i) 3 (ii) $(n+3)r^n$ (iii) $nr^{n-2}(\mathbf{r} \cdot \mathbf{c})$
3. (i) $x^2\hat{\mathbf{i}} - 2xy\hat{\mathbf{j}}$ (ii) 0 (iii) $-2\mathbf{c}$
4. (i) $h_r = 1, h_\theta = r, h_z = 1$ (ii) $h_r = 1, h_\phi = r \sin\theta, h_\theta = r$ (iii) $h_1 = h_2 = (u^2 + v^2)^{1/2}, h_3 = uv$
5. $\frac{12}{5}\pi R^5$
6. $\frac{3}{4}\pi a^4$
7. $\frac{4}{5}\pi a^5\hat{\mathbf{k}}$
8. $2\pi a^2 h^2$