2A1C Vector Algebra and Calculus I

Bugs/queries to ian.reid@eng.ox.ac.uk
Course page www.robots.ox.ac.uk/∼ian/Teaching/Vectors
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Ian Reid

1. The tetrahedron in the figure has vertices A, B, C, D at positions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, respectively.

   (a) Derive an expression for the unit normal $\hat{\mathbf{n}}_{abc}$ to the planar face ABC.

   Explain why points $\mathbf{r}$ on the plane ABC can be written $\mathbf{r} \cdot \hat{\mathbf{n}}_{abc} = \mathbf{a} \cdot \hat{\mathbf{n}}_{abc}$.

   Hence, and by writing $\mathbf{y} = \mathbf{d} + \beta \hat{\mathbf{n}}_{abc}$, derive the position of Y, the projection of D onto the face ABC along the normal to ABC.

   (b) By writing $\mathbf{x} = \mathbf{a} + \alpha(\mathbf{c} - \mathbf{a})$, derive an expression for the position vector $\mathbf{x}$ of the point X on the line AC such that DX is perpendicular to AC.

   (c) The unit normal you derived in part (a) might point into, or out of, the tetrahedron. With knowledge of the ordering of vertices around each face (say from the diagram) it is of course possible to recover an outward facing normal.

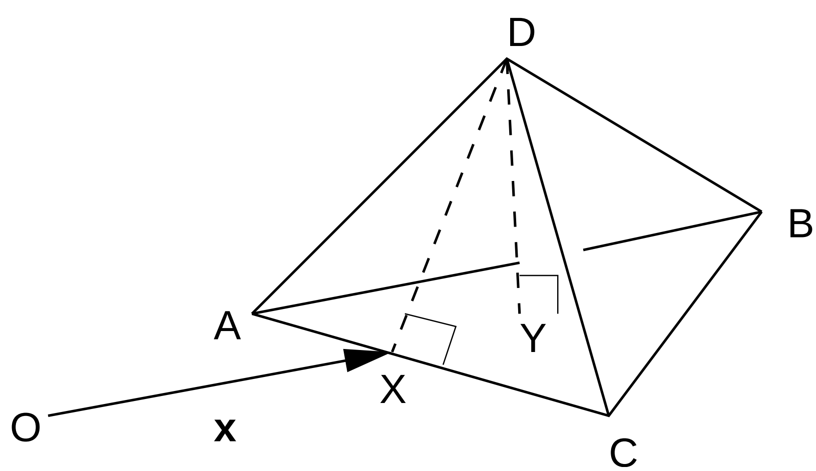
   Now suppose that you do not know the ordering.

   (i) By considering the vector from D to A, devise a procedure to check whether the $\hat{\mathbf{n}}_{abc}$ you recovered in part (a) is a unit normal pointing out of the tetrahedron.

   (ii) Would using the "wrong" sign for a normal affect the result of part (a)?

   (d) In terms of outwards facing unit normals, derive an expression for the internal angle between faces ABC and BCD.

   Would using the "wrong" sign for a normal affect the result of part (d)?



2. The figure shows a laser mounted on a pan-tilt mechanism so that its beam can be directed from the point $\mathbf{a}$ in any direction $\hat{\mathbf{b}}$. The beam must be reflected from a plane mirror with normal $\hat{\mathbf{n}}$ facing out of the mirror to hit a target lying at position $\mathbf{t}$.

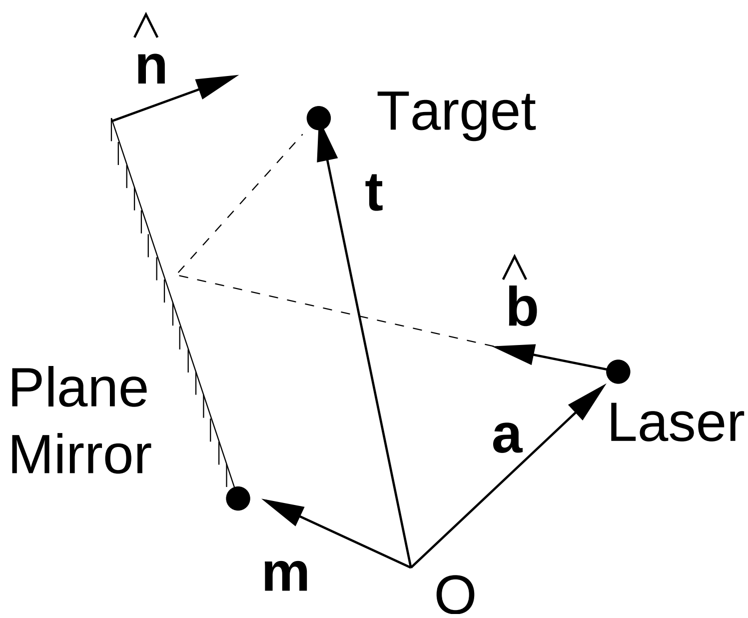
   (a) Derive the equation of the mirror plane.

   (b) Derive an expression for the minimum straight-line distance of a point $\mathbf{t}$ from the plane mirror.

   (c) Show that the direction the laser must point in is $\hat{\mathbf{b}} = \mathbf{v}/|\mathbf{v}|$ where

   $$\mathbf{v} = \mathbf{t} - 2\left((\mathbf{t} - \mathbf{m}) \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}} - \mathbf{a} \quad .$$

   (d) A valid target position $\mathbf{t}$ must lie in front of the mirror. Devise a vector test for this case.



3. The figure shows two ground-based radar stations tracking an aircraft. Station 1 has a fixed Cartesian coordinate system $(Oxyz)_1$ attached to it, and measures the direction to an aircraft as $\hat{\mathbf{n}}_1$, but it cannot determine the range ρ_1. Station 2 is located at $\mathbf{a}$ in the first's coordinate system. In its local coordinate system $(Oxyz)_2$ it simultaneously measures the direction to the aircraft as $\hat{\mathbf{n}}_2$ but again does not determine the range ρ_2.

   (a) Assume that the two local coordinate systems are aligned (ie there is no rotation between them). Show that the aircraft's position in $(Oxyz)_1$ can be written in two ways, as either $\mathbf{p}_1 = \rho_1\hat{\mathbf{n}}_1$ or as $\mathbf{p}_1 = \mathbf{a} + \rho_2\hat{\mathbf{n}}_2$. Hence show that the range from the first station is

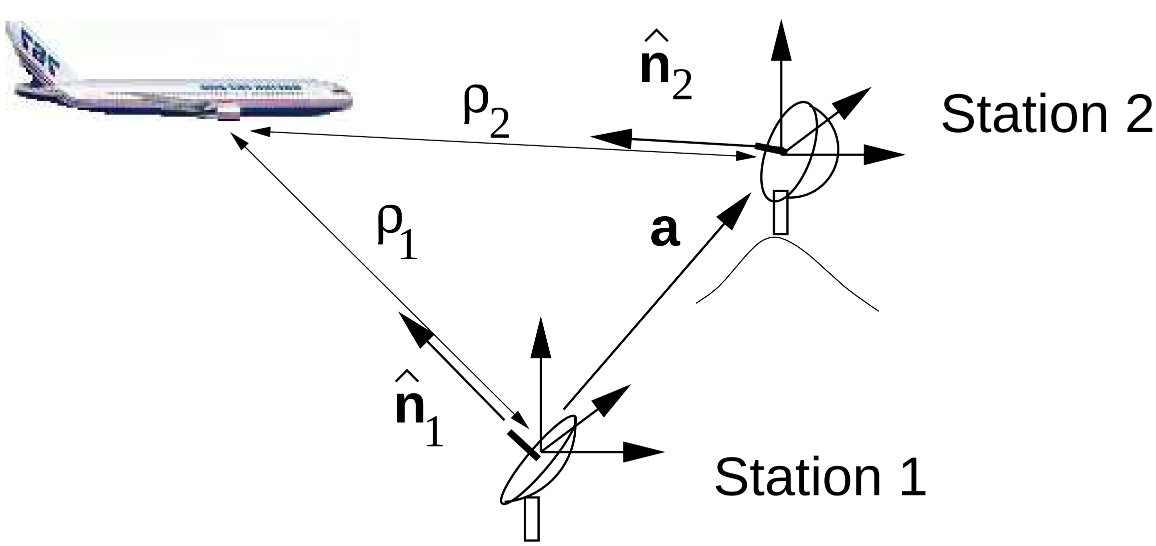
$$\rho_1 = \frac{(\mathbf{a} \times \hat{\mathbf{n}}_2) \cdot (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)}{|\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2|^2}$$

   (b) The aircraft is at $\mathbf{p}_1 = [-20, 5, 5]$ km relative to $(Oxyz)_1$, and $\mathbf{a} = [1, 10, 0.3]$ km. Use a vector magnitude to derive the range ρ_1, and deduce $\hat{\mathbf{n}}_1$. Also derive the position $\mathbf{p}_2$ of the aircraft as measured in $(Oxyz)_2$, and hence find $\hat{\mathbf{n}}_2$.

   (c) Verify the formula derived in part (a) by inserting the values of $\hat{\mathbf{n}}_{1,2}$ found in part (b).

   (d) What simple change to the expression for ρ_1 would you have to make if the two coordinate systems were still separated by $\mathbf{a}$ but were no longer aligned?

   (e) Under what circumstances would the expression of part (a) break down? Relate your answers to the physical situation.



4. (a) The vector triple-product is $\mathbf{p} \times (\mathbf{q} \times \mathbf{r}) = (\mathbf{p} \cdot \mathbf{r})\mathbf{q} - (\mathbf{p} \cdot \mathbf{q})\mathbf{r}$.
   Starting from the cyclic property of the scalar triple-product $(\mathbf{p} \times \mathbf{q}) \cdot \mathbf{r} = (\mathbf{r} \times \mathbf{p}) \cdot \mathbf{q}$, use judicious substitution of vectors to
   i. Express $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$ in terms of scalar products only; and hence
   ii. Show that $(\mathbf{a} \times \mathbf{b}) \cdot ((\mathbf{a} \times \mathbf{c}) \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{d})(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}))$.

   (b) i. Express $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$ as a linear combination of $\mathbf{c}$ and $\mathbf{d}$.
   ii. By writing the expression in square brackets as a linear combination of $\mathbf{a}$ and $\mathbf{c}$, simplify $(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})]$.

   (c) By writing $\mathbf{x} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu(\mathbf{a} \times \mathbf{b})$ (where $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors, and where λ, μ and ν are scalars), derive general solutions to the following equations
   i. $\mathbf{x} \times \mathbf{a} = \mathbf{b}$, given that $\mathbf{a} \cdot \mathbf{b} = 0$.
   ii. $\mathbf{x} \cdot \mathbf{a} = \gamma$.

5. A fairground ride comprises a disc in the horizontal xy-plane, its centre at the origin of the world coordinate system. A track runs from the centre to the edge of the disk and is initially coincident with the world's $\hat{\mathbf{\imath}}$ axis. At time $t = 0$ (i) a car moves off along the track from the centre of the disk with constant speed ν, and (ii) the disk is set to rotate with angular velocity $\omega\hat{\mathbf{k}}$ about its centre.

   (a) Draw a diagram and confirm that in the world coordinate system the car's position at time t is $\mathbf{R}(t) = \nu t\hat{\mathbf{r}}(t)$, and deduce an expression for $\hat{\mathbf{r}}(t)$.
   (b) Show that the car's velocity in the world frame is $\mathrm{d}\mathbf{R}/\mathrm{d}t = \nu\hat{\mathbf{r}}(t) + \nu t\omega\hat{\boldsymbol{\theta}}(t)$.
   (c) Derive the car's acceleration in the world frame, and interpret the terms you find.
   (d) Repeat steps (a) - (c) for the case where the car moves off along the track with constant acceleration α. Again discuss the terms present in the car's acceleration.

6. The mutually perpendicular unit vectors $\hat{\mathbf{l}}, \hat{\mathbf{m}}, \hat{\mathbf{n}}$ are functions of time t, but remain mutually perpendicular unit vectors. As $\hat{\mathbf{l}}, \hat{\mathbf{m}}, \hat{\mathbf{n}}$ form a basis, it must be the case that $\mathrm{d}\hat{\mathbf{l}}/\mathrm{d}t = \alpha_1\hat{\mathbf{l}} + \beta_1\hat{\mathbf{m}} + \gamma_1\hat{\mathbf{n}}$, where α_1 etc are (as yet) arbitrary coefficients; and similarly for $\mathrm{d}\hat{\mathbf{m}}/\mathrm{d}t$ and $\mathrm{d}\hat{\mathbf{n}}/\mathrm{d}t$.

   (a) By differentating $\hat{\mathbf{l}} \cdot \hat{\mathbf{l}} = 1$ w.r.t. time, and similarly for $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$, show that

   $$\mathrm{d}\hat{\mathbf{l}}/\mathrm{d}t = \beta_1\hat{\mathbf{m}} + \gamma_1\hat{\mathbf{n}} \qquad \mathrm{d}\hat{\mathbf{m}}/\mathrm{d}t = \alpha_2\hat{\mathbf{l}} + \gamma_2\hat{\mathbf{n}} \qquad \mathrm{d}\hat{\mathbf{n}}/\mathrm{d}t = \alpha_3\hat{\mathbf{l}} + \beta_3\hat{\mathbf{m}}$$

   (b) By differentiating $\hat{\mathbf{l}} \cdot \hat{\mathbf{m}} = 0$, and similarly for the other vectors, obtain three relationships of the form $\alpha_2 + \beta_1 = 0$. Hence, and using the vector test for coplanarity, show that $\mathrm{d}\hat{\mathbf{l}}/\mathrm{d}t$, $\mathrm{d}\hat{\mathbf{m}}/\mathrm{d}t$, and $\mathrm{d}\hat{\mathbf{n}}/\mathrm{d}t$, are coplanar.

   (c) Consider physically how three mutually perpendicular unit direction vectors can change, and so justify this coplanarity result.

   (d) Confirm that your expressions obtained earlier are verified when you differentiate the expression $\hat{\mathbf{l}} \times \hat{\mathbf{m}} = \hat{\mathbf{n}}$ with respect to t.

7. The axis of a helical tube with cross-sectional area A is given by $\mathbf{r}(p) = [a\sin p, a\cos p, bp]$. The tube carries an inviscid, incompressible fluid at a volume flow rate of L units per second.

   (a) Find a unit tangent vector $\hat{\mathbf{t}}$ to the axis, and hence determine p in terms of arc-length, s.

   (b) By determining the relationship between arc-length s and time t, write down the instantaneous velocity vector of an element of fluid on the axis.

   (c) Find $\mathrm{d}\hat{\mathbf{t}}/\mathrm{d}s$ and the curvature of the axis. Hence determine the direction and magnitude of the instantaneous acceleration of an element of fluid on the axis. Discuss the signs of the components of this acceleration vector.

8. (a) A scalar potential field is given by ϕ and the vector field $\mathbf{F}$ is

$$\mathbf{F} = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z} \ .$$

Noting that $\mathrm{d}\mathbf{r} = \hat{\imath}\mathrm{d}x + \hat{\jmath}\mathrm{d}y + \hat{\mathbf{k}}\mathrm{d}z$, and recalling the definition of a perfect or total differential, show that the line integral

$$\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_{\mathrm{A}}^{\mathrm{B}} \mathrm{d}\phi = \phi_{\mathrm{B}} - \phi_{\mathrm{A}} \ .$$

(b) The vector field $\mathbf{F} = yz\hat{\imath} + xz\hat{\jmath} + xy\hat{\mathbf{k}}$. Evaluate the line integral $\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{F}\cdot\mathrm{d}\mathbf{r}$, where A is $(0,0,0)$ and B is $(4,2,4)$, in two ways:

i. by integrating to find that a potential function ϕ exists, and evaluating the difference in potential between the points.

ii. by integration along the parametrized curve $x = p^2, \quad y = p, \quad z = 2p$. To do this, express $\mathbf{F}$ in terms of p, and $\mathrm{d}\mathbf{r}$ in terms of p and $\mathrm{d}p$.

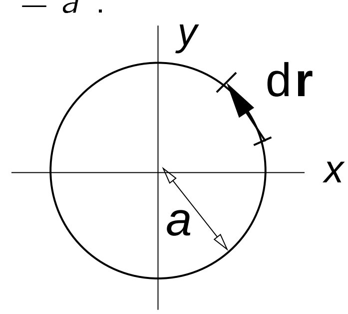
9. (a) Write down **vector** expressions for the elements of line $\mathrm{d}\mathbf{r}$ and surface $\mathrm{d}\mathbf{S}$ and **scalar** expressions for the elements of volume $\mathrm{d}V$ shown in the figure, using the appropriate coordinate system.

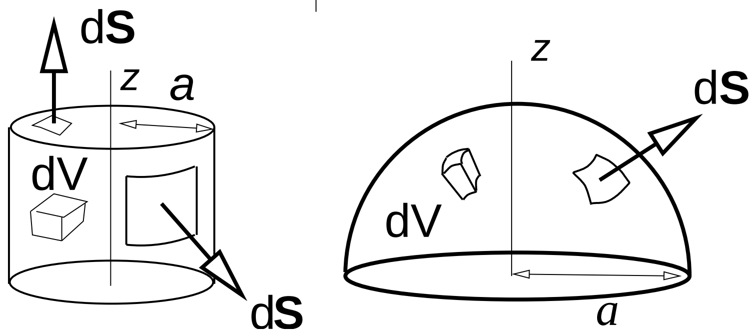
(b) Find the work done $\int \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ in a field $\mathbf{F}(x,y) = y\hat{\imath} + x\hat{\jmath}$ when moving along

i. the shorter circular path from $(x,y) = (a,0)$ to $(x,y) = (a/\sqrt{2}, a/\sqrt{2})$;

ii. the longer path from $(x,y) = (a/\sqrt{2}, a/\sqrt{2})$ to $(x,y) = (a,0)$.

(c) Find $\int_S \mathbf{v}\cdot\mathrm{d}\mathbf{S}$ where $\mathbf{v} = r^2(x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}})$ and S is the *entire* surface of the sphere $x^2 + y^2 + z^2 = a^2$.





Answers and Hints

1. (a) $\mathbf{y} = \mathbf{d} + ((\mathbf{a} - \mathbf{d}) \cdot \hat{\mathbf{n}}_{abc})\hat{\mathbf{n}}_{abc}$
   (b) $\alpha = (\mathbf{d} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a})/|\mathbf{c} - \mathbf{a}|^2$

2. (b) $\alpha = (\mathbf{t} - \mathbf{m}) \cdot \hat{\mathbf{n}}$
   (c) Think where the image of $\mathbf{t}$ appears ...

3. (a) Think about a vector operation to eliminate ρ_2 from the equation.
   (b) Using $\mathbf{p}$ alone, $\rho_1 = \sqrt{450}$ km
   $\hat{\mathbf{n}}_1 = [-0.942,\ 0.236,\ 0.236]$, $\hat{\mathbf{n}}_2 = [-0.9505,\ -0.2263,\ 0.2127]$

   (c) Intermediate checks ...
   $\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2 = [0.1035,\ -0.0235,\ 0.4374]$, $\mathbf{a} \times \hat{\mathbf{n}}_2 = [2.1953,\ \ -0.4979,\ \ 9.2791]$.

4. (a) $(\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$ then use replacement to continue.
   (b) $[\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]^2$ or cyclic permutations thereof.
   (c) $\mathbf{x} = \lambda\mathbf{a} + \mu\mathbf{b} + \gamma\mathbf{a} \times \mathbf{b}$, where
      (i) λ arbitrary, $\mu = 0$, and $\nu = 1/a^2$
      (ii) λ arbitrary, $\mu = (\gamma - \lambda a^2)/\mathbf{a} \cdot \mathbf{b}$, and ν arbitrary.

5. (d) $\ddot{\mathbf{R}} = \alpha(1 - t^2\omega^2/2)\hat{\mathbf{r}} + 2\alpha t\omega\hat{\boldsymbol{\theta}}$

6. (d) Hint: differentiate both sides, and then use parts (a) and (b) to show that they are the same.

7. (a) $\hat{\mathbf{t}}(p) = (a^2 + b^2)^{-1/2}[a\cos p, -a\sin p, b]$, $p = (a^2 + b^2)^{-1/2}s$; (b) $\mathrm{d}s/\mathrm{d}t = L/A, \mathbf{v}(p) = L/A\hat{\mathbf{t}}(p)$; (c) $\kappa = a/(a^2 + b^2), \hat{\mathbf{n}} = [-\sin p, -\cos p, 0], \mathrm{d}\mathbf{v}/\mathrm{d}t = (L^2/A^2)\kappa\hat{\mathbf{n}}$.

8. (b) i) 32 and (ii) 32

9. (a) Consult the lecture notes
   (b) $a^2/2$ and $-a^2/2$. Think why they sum to zero.
   (c) $4\pi a^5$