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Hilary Term 2003

APPLIED ESTIMATION II

As in the first set of exercises, you may find some of the questions are most easily answered using **matlab**. Code examples from the lectures are available from my web-site at

http://www.robots.ox.ac.uk/∼ian/Teaching/Estimation/

1. *Kalman filtering properties.*

   Consider the one-dimensional system model described by,

$$\begin{aligned} x(k+1) &= F(k)x(k) + w(k) &(1)\\ z(k+1) &= x(k+1) + v(k+1) &(2)\end{aligned}$$

   Suppose also that,

$$E[w^2(k+1)] = q, \qquad E[v^2(k+1)] = r, \qquad \text{for all } k.$$

   (a) Show that the predicted state covariance satisfies,

$$P(k+1|k) \geq q.$$

   (b) Show that the estimated state covariance satisfies,

$$P(k+1|k+1) \leq r \quad \text{and} \quad P(k+1|k+1) \geq \frac{rq}{r+q}.$$

   (c) Show that the Kalman gain satisfies,

$$0 \leq K(k+1) \leq 1.$$

   What do these results tell you about how initialisation conditions relate to prediction and estimation accuracy?

2. *Understanding the Kalman filter*

   The Kalman filter update equations based on the innovation are,

$$\begin{aligned} \hat{\mathbf{x}}_{k+1|k+1} &= \hat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}\boldsymbol{\nu}_{k+1}, \\ \mathbf{P}_{k+1|k+1} &= \mathbf{P}_{k+1|k} - \mathbf{K}_{k+1}\mathbf{S}_{k+1}\mathbf{K}_{k+1}^T \end{aligned}$$

   (a) Use these equations to deduce that the gain matrix $\mathbf{K}_{k+1}$ that minimises mean squared error is given by

$$\mathbf{K}_{k+1} = \mathbf{P}_{k+1|k}\mathbf{H}_{k+1}^T\mathbf{S}_{k+1}^{-1}$$

(b) Explain the meaning of the term $\mathbf{H}_{k+1}^T \mathbf{S}_{k+1}^{-1}$.

(c) Explain how these equations are modified if observation and process noises are correlated,

$$E[\mathbf{v}_k \mathbf{w}_k] = \mathbf{C}_k.$$

3. *Continuous and discrete state-space models.*

Consider a vehicle tracking problem where a vehicle moves at constant acceleration and measurements are made of vehicle position. Let $x(t), \dot{x}(t)$ and $\ddot{x}(t)$ represent the vehicle position,velocity and acceleration, and $\mathbf{w}(t)$ represent the uncertainty in the process model. Further let the measurement of vehicle position $y(t)$ be corrupted by measurement noise $v(t)$.

(a) Write down the continuous-time domain system state and observation equations.

(b) Using your answer to part (1) derive from first principles the equivalent discrete-time system and measurement equations.

(c) In practice, what conditions must be met if the discrete-model of part (2) is to be a good representation of the continuous-model of part (1)?

4. *State-space models of nonlinear systems.*

A simple discrete-time model of car motion is:

$$\begin{bmatrix} x(k+1) \\ y(k+1) \\ \theta(k+1) \end{bmatrix} = \begin{bmatrix} x(k) + \triangle T V(k) cos\theta(k) \\ y(k) + \triangle T V(k) sin\theta(k) \\ \theta(k) + \triangle T \phi(k) \end{bmatrix}, \tag{3}$$

where x, y, θ are the car location and orientation respectively, and $V(k+1), \phi(k+1)$ are the forward velocity and steering angle of the car. Assume that $V(k+1)$ and $\phi(k+1)$ are known perfectly. As the car drives it is subject to random disturbances so that the true location differs from that which would be computed from knowledge of the velocity and steering angle. Linearise the model (i.e. derive a first order Taylor series approximation), and use this to obtain (in state-space form) a linear differential equation describing the evolution of the error between true location and the location computed from $V(k)$ and $\phi(k)$.

5. *Steady state performance*

(a) Explain why the gain matrix $K(k+1)$ tends to a steady-state value over time.

(b) Consider the scalar discrete-time linear system and observation model

$$x(k+1) = x(k) + w(k); \qquad z(k+1) = x(k+1) + v(k+1),$$

where the noise terms $w(k)$ and $v(k)$ are uncorrelated in time, with

$$E[w^2(k)] = q = 0.01; \qquad E[v^2(k)] = r = 0.1.$$

Compute the steady-state estimate covariance for the Kalman filter algorithm with these system and observation models. Hence also compute the steady-state gain.

(c) Write down the steady-state update equation for the system in part (b). Show that this may simply be viewed as a low-pass filter, and explain the role of q and r in determining the cut-off frequency of this filter.

6. *Constant acceleration particle system.*

   A car is tethered into a flat-bed truck which initially stands at rest in a loading bay. The truck then moves along a straight track with unit acceleration. The longitudinal position of the car relative to the truck is governed by a Gaussian noise process with unit variance. The distance of the car from a fixed point in the loading bay is measured, at successive time steps, by a radar sensor with Gaussian measurement noise and covariance σ^2.

   (a) Derive a Kalman filter to estimate the position and velocity of the car, relative to the loading bay, at successive time steps.

   (b) Show that, in the case that $\sigma = 1$, the Kalman gain reaches a steady state value of $K_\infty = (\sqrt{5} - 1)/2$.

   (c) What test could you apply to the innovation to indicate whether the acceleration of the truck had changed?

7. *Filter performance - innovation test*

   (a) The three main tests for the correct operation of a Kalman filter are:

      i. that 95% of the innovations have magnitude less than $2\sqrt{S(k+1)}$.
      ii. that the mean of the normalised innovation sequence tends to a fixed specified value within a fixed confidence bound.
      iii. that the time-averaged auto-correlation of the innovation sequence is zero everywhere except at the origin.

      Explain why each of these must be true and show how they are tested in practice.

   (b) Figure 1 is derived from the results of running a Kalman filter algorithm on a sequence of measurements. Explain, giving a reasoned argument, what is wrong with the filter.

8. *Extended Kalman Filter*

   In what, if any, sense is the extended Kalman filter "optimal"?

   A particle is moving with constant velocity along the line $y = 20.$ with approximately uniform velocity subject to a velocity distribution with mean zero and strength q. It is observed using a bearing-only sensor located at the origin. Assume that the observation noise is σ_θ^2.

   (a) Write down the process and observation equations for this system.

   (b) Find an expression for the extended Kalman filter gain matrix as a function of state prediction and prediction covariance.

   (c) Plot the gain matrix as a function of predicted target bearing. Explain the variation in gain matrix and the effect of low-horizon bearing on estimation accuracy.

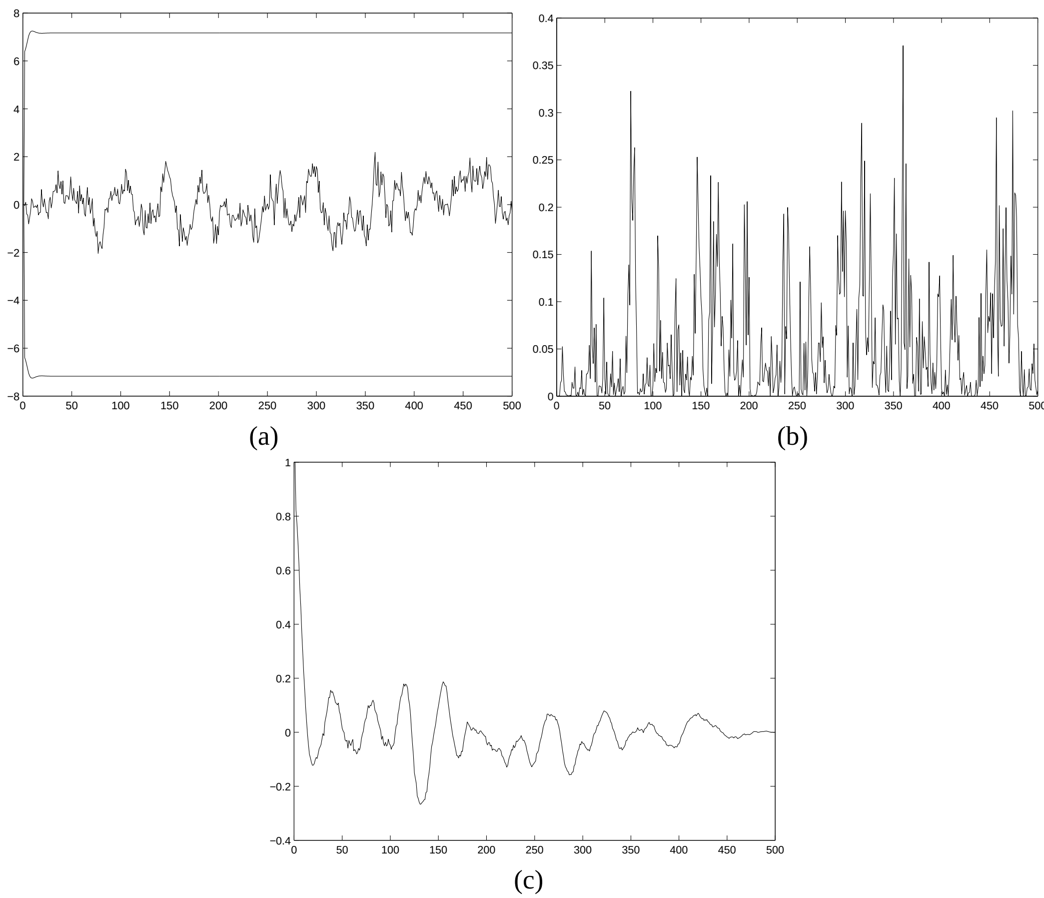

Figure 1: Question 7: (a) innovation; (b) normalised innovations squared; (c) autocorrelation of the innovation.