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APPLIED ESTIMATION I

1. Joint probabilities for the discrete random variables X, Z are given by

	$X = 1$	$X = 2$	$X = 3$
$Z = 1$	0.2	0.07	0.06
$Z = 2$	0.09	0.1	0.01
$Z = 3$	0.02	0.12	0.09
$Z = 4$	0.06	0.08	0.1

   (a) Check the normalisation of the joint distribution
   (b) Compute the marginal probability distributions for X, Z.
   (c) Compute the conditional distribution $P(Z|X = 3)$
   (d) Use Bayes' rule to compute $P(X = 3|Z = 4)$ using only information in the previous 2 parts; then check the answer by direct calculation from the table above.

2. (a) If the independent continuous variables x_1, x_2 are normally distributed with means and variances $\mu_i, \sigma_i^2,\ i = 1, 2$, derive the distribution of their sum $x_1 + x_2$ (*Hint: Consider the moment generating function (Characteristic function) of the random variables*)
   (b) Now suppose that those variables are no longer independent but have correlation coefficient ρ. Find the covariance and information matrices of the vector variable $\mathbf{x} = (x_1, x_2)^T$. Write down the joint density function *including* the normalisation constant.

3. (a) Find the conditional density function $p(x_2|x_1)$ for the problem above, including the normalisation constant.
   (b) What are the conditional mean $\mathcal{E}[x_2|x_1]$ and the conditional variance $\mathcal{V}[x_2|x_1]$?

4. The data of question 1 could be taken as characterising a noisy sensor with output Z, intended to measure a state X.
   (a) Given that a measurement $Z = 3$ has been taken, what is the MLE for X?
   (b) Now, if conditional probabilities $P(Z = i|X = j)$ are as in part a) but prior probabilities are as follows:

$$P(X = 1) = 0.2,\ P(X = 2) = 0.5,\ P(X = 3) = 0.3$$

   find the MAP estimate for X.

5. Independent sensors have outputs $z_1, z_2,\ i = 1, 2$ measuring a state x which are unbiased and normally distributed with standard deviation $\sigma_i,\ i = 1, 2$.
   (a) Derive the maximum likelihood estimate of x given single measurements z_1, z_2 from each sensor.

(b) If also the state itself is known to have a prior distribution that is normal $\mathcal{N}(\overline{x}, \sigma_0^2)$ what is the posterior density for x?

(c) If, instead, the prior density is given by the exponential distribution

$$p(x) = \begin{cases} a \exp(-ax) & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and only a single sensor reading z_1 is made, what is the MAP estimate for x?

(d) Suppose in part b) that

$$\overline{x} = 10, \ \sigma_0 = 0.3, \ \ \sigma_1 = \sigma_2 = 0.4$$

and that the sensor readings are

$$z_1 = 10.5 \quad z_2 = 12.0.$$

What is the MAP estimate for x, based on these data?

(e) Apply validation tests to z_1 and z_2 and show that one sensor fails and the other passes.

(f) What now is the MAP estimate for x in b), given only the validated sensor reading?

6. Consider a surveillance system that uses a video camera to track vehicle travelling down a steep ramp into a car park. Vehicle position at timestep n is $\mathbf{x}_n$cm and from the $n-1$th to the nth time step the position of a vehicle is assumed to change by a random displacement whose mean is $(0, 0.1n)^T$ and whose variance is

$$\begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.3 \end{pmatrix}$$

(a) If the initial mean and covariance of the vehicle's position (at time $n = 0$) are

$$\begin{pmatrix} 100 \\ 100 \end{pmatrix} \text{ and } \begin{pmatrix} 10 & 10 \\ 10 & 20 \end{pmatrix}$$

what is the probability distribution after $n = 100$ time-steps?

(b) Now a two-dimensional measurement $\mathbf{z}_1$ of vehicle position is made that is unbiased and with a random error that is normal and isotropic — ie such that its standard deviation in any direction is constant — and the value of that s.d. is 10 cm. If $\mathbf{z}_1 = (90, 600)^T$ what now is the posterior distribution for the vehicle's position?

(c) After a further 10 timesteps, a second measurement is to be made. What is the effective prior density for $\mathbf{x}$, just before making the new measurement?

(d) The second measurement $z_2 = 700$ is made by a different sensor that is one-dimensional, measuring distance in the y direction only. It is a biased sensor that under-reads by 4cm and has random error with s.d. 5cm. What now is the posterior density of the position?

(e) Write a **matlab** function that would plot each of these distributions as covariance ellipses (and plot them, if you like).

7. Data

$$(X_n, Y_n) = (0, 0), \ (1, 1), \ (2, 1.8), \ (3, 2.5)$$

is given in which the variance associated with the Y_n is

$$\sigma_n^2 = \sigma^2,$$

constant for all n.

(a) Use the recursive least-squares algorithm to fit a linear function to the data.

(b) Compare your result with the "batch" algorithm that uses the pseudo-inverse of the "Vandermonde" matrix (see Engineering Computation 2nd year notes).

Answers

2. b) $p(\mathbf{X}) = K^{-1} \exp -\frac{1}{2}(\mathbf{x} - \bar{\mathbf{x}})^T S(\mathbf{x} - \bar{\mathbf{x}})$ with $K = 2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}$ and

$$S = \frac{1}{1-\rho^2} \begin{pmatrix} 1/\sigma_1^2 & -\rho/\sigma_1\sigma_2 \\ -\rho/\sigma_1\sigma_2 & 1/\sigma_2^2 \end{pmatrix}$$

3. a) $x_2|x_1 \sim \mathcal{N}(\bar{x}_2, \sigma_2')$ with

$$\bar{x}_2 = \mu_2 + \frac{\sigma_2}{\sigma_1}\rho(x_1 - \mu_1) \text{ and } \sigma_2' = \sigma_2\sqrt{1-\rho^2}.$$

4. a) $\hat{X} = 3$; b) $\hat{X} = 2$.

5. c)

$$\hat{x} = \begin{cases} z_1 - a\sigma_1^2 & \text{if } z_1 \geq a\sigma_1^2 \\ 0 & \text{otherwise} \end{cases}$$

6.

a) Normal $\mathcal{N}(\bar{\mathbf{x}}, \bar{P})$ with

$$\bar{\mathbf{x}} = \begin{pmatrix} 100 \\ 605 \end{pmatrix}, \quad \bar{P} = \begin{pmatrix} 20 & 20 \\ 20 & 50 \end{pmatrix}$$

b) Normal $\mathcal{N}(\hat{\mathbf{x}}, P)$ with

$$\hat{\mathbf{x}} = \begin{pmatrix} 97.95 \\ 602.27 \end{pmatrix}, \quad P = \begin{pmatrix} 14.8 & 11.4 \\ 11.4 & 31.8 \end{pmatrix}$$

c) Normal $\mathcal{N}(\bar{\mathbf{x}}', \bar{P}')$ with

$$\bar{\mathbf{x}}' = \begin{pmatrix} 97.95 \\ 707.77 \end{pmatrix}, \quad \bar{P} = \begin{pmatrix} 15.8 & 12.4 \\ 12.4 & 34.8 \end{pmatrix}$$

d) Normal $\mathcal{N}(\hat{\mathbf{x}}', P')$ with

$$\hat{\mathbf{x}}' = \begin{pmatrix} 97.17 \\ 705.57 \end{pmatrix}, \quad P' = \begin{pmatrix} 13.21 & 5.17 \\ 5.17 & 14.55 \end{pmatrix}$$

7. a) $Y = 0.08 + 0.83X$.