Computational Geometry

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Hints etc will be be placed on the web at
http://www.robots.ox.ac.uk/∼ian/Teaching/CompGeom/
as will pdf of the sheet.

1. The graphics model of an intergalactic battlecruiser is described by points $\mathbf{X}_M$ defined in a coordinate system M which is rotated and translated from a reference coordinate frame W such that, in inhomogeneous coordinates,

$$\mathbf{X}_M = \mathtt{R}_{MW}\mathbf{X}_W + \mathbf{T}_M \ .$$

The scene is viewed by a graphics camera with

$$\mathbf{X}_C = \mathtt{R}_{CW}\mathbf{X}_W + \mathbf{T}_C \ .$$

(a) Derive an expression for 4×4 homogeneous transformation $\mathtt{E}$, which transforms points in the model frame into points in the camera frame, $\mathbf{X}_C = \mathtt{E}\mathbf{X}_M$.

(b) The reference-model rotation and translation are

$$\mathtt{R}_{MW} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \mathbf{T}_M = (-4,\ 0,\ 0)^\top$$

You also know that the origin of C frame is at $\mathbf{X}_W = (0, 1, -2)^\top$, that Z_C points towards the origin of the reference coordinate frame, and that X_C is parallel to X_W.

i. Sketch the coordinate frames.

ii. Evaluate the elements of the matrix $\mathtt{E}$.

(c) The graphics camera has its centre of projection at the origin of C and forms perspective images on the plane $Z_C = 1/2$. At what position is the origin of the model frame imaged?

2. (a) i. Show that any two triangles can be put in affine correspondence.

ii. Three points, $(x, y) = (0, 0), (1, 0), (0, 1)$, correspond under an affine transformation to $(x', y') = (0, 0), (1, 0), (1, 1)$.
Derive the homogeneous affine transformation $\mathtt{A}$, and compute the point corresponding to $(x, y) = (1, 1)$.

(b) Show that under an affine plane-to-plane transformation

i. Parallel lines transform to parallel lines;

ii. The centre of a set of points transforms to the centre of the transformed set;

iii. Ratios of lengths of parallel line segments are invariant (unchanged);

iv. Ratios of areas of two triangles are invariant. (Hint: express the area of a general triangle as a determinant.)

3. (a) Show using homogeneous coordinates that, under perspective projection, parallel lines in the scene meet at a vanishing point in the image.

   (b) The battlecruiser in Q1 fires two photon torpedoes from points $\mathbf{X}_M = (0,0,1)$ and $(0,0,-1)$ on the model, both in the model direction $\hat{\mathbf{D}}_M = (1,0,0)$. They miss their target, and whizz off into deep space glowing a delicate purple.

   Determine the start and end points of the two lines that the graphics animator will cause to be drawn in the image.

4. An advertiser wants to paint an advert on the outfield at Lords' so that it appears upright in the image of a camera behind the bowler's arm. First, he draws a unit square on the ground, and measures its position in the camera's 640 by 480 pixel image. The unit square on the world plane:

$$(x,y) = (0,0),\ (1,0),\ (1,1)\ (0,1)$$

is mapped, by projection, onto the image plane as

$$(x',y') = (240,400),\ (400,400),\ (360,320),\ (280,320)$$

   (a) Determine the homogeneous matrix that describes the mapping $\mathbf{x}' = \mathtt{H}\mathbf{x}$, using Matlab.

   (b) Assuming he would like the advert to appear in the image as a rectangle 200 pixels wide, by 80 pixels high, with top-left coordinate at (220,320), determine the coordinates of the corners of the area on the ground in which he should paint.

5. Use homogeneous transformations to find the type, dimensions, orientation and location of the quadric $\mathbf{X}^\top \mathtt{Q}\mathbf{X} = 0$ where

$$\mathtt{Q} = \begin{pmatrix} 10 & -10 & 0 & -1 \\ -10 & 10 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & -17 \end{pmatrix}$$

6. (a) By parametrizing the cylinder

$$(x-2)^2 + y^2 = 4$$

   find an explicit parametric form for its curve of intersection with the ellipsoid

$$(x+y)^2 + \frac{(x-y)^2}{4} + z^2 = 1.$$

   (b) Now repeat that, replacing the cylinder with a cone, apex at the origin, semi-angle α and axis in the (1,0,1) direction.

7. (a) Compute, for the y-component only, the a_i, b_i, c_i, d_i values for the four natural cubic spline sections $i = 0, \ldots, 3$ which interpolate the 5 points $\mathbf{x}_0 = (0,0)$, $\mathbf{x}_1 = (1,0)$, $\mathbf{x}_2 = (1,1)$, $\mathbf{x}_3 = (0,1)$. $\mathbf{x}_4 = (0,2)$.

   (b) Show that in moving from degree N to degree $N+1$, the control points $\mathbf{x}'$ that preserve the shape of a Bezier curve are related to the degree N control points $\mathbf{x}$ by

$$\begin{aligned} \mathbf{x}'_0 &= \mathbf{x}_0 \\ \mathbf{x}'_i &= \frac{i}{N+1}\mathbf{x}_{i-1} + \left(1 - \frac{i}{N+1}\right)\mathbf{x}_i \quad i = 1, \ldots, N \\ \mathbf{x}'_{N+1} &= \mathbf{x}_N \end{aligned}$$

8. A 3D tensor-product **quadratic** Bezier patch is set up with control points at

$$\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

   (a) What is the direction of the surface normal at the point $(-1,-1,0)^\top$?

   (b) Find the "extent" (cuboid with faces perpendicular to the x, y, z axes respectively) that fits as closely as possible around the patch.

9. (a) Invent an algorithm for finding whether two convex polygons intersect. What is the *computational complexity* of the algorithm?

   (b) How might you find the convex hull of a 3D object?

Answers

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If you have trouble do email me, and I will place the reply on the web.

1. The 4×4 matrix such that $\mathbf{X}_C = \mathtt{E}_{CW}\mathbf{X}_W$ is

$$\mathtt{E}_{CW} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & -1/\sqrt{5} & 2/\sqrt{5} & \sqrt{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the whole transform from model to camera frames is

$$\mathtt{E} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 & 4/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} & 0 & 13\sqrt{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence origin of model frame at $(0, 4/\sqrt{5}, 13/\sqrt{5}, 1)^\top$ in camera frame, and imaged at $(x, y) = (0, 2/13)$.

2. $(x', y') = (2, 1)$.

3. Start points $\left(\frac{\pm\sqrt{5}}{26}, \frac{2}{13}\right)^\top$. VP at $\left(0, \frac{1}{4}\right)^\top$

4. (a) Up to scale

$$\mathtt{H} = \begin{bmatrix} 0.2520 & 0.5040 & 0.3780 \\ 0.0000 & 0.3780 & 0.6299 \\ 0 & 0.0016 & 0.0016 \end{bmatrix}$$

(b) $(x, y) = (-0.125, 0), (1.125, 0), (1.75, 1), (-0.75, 1)$

8. $|x| \le 1$, $|y| \le 1$, $0 \le z \le 1/4$.