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# Parameterized Analysis of Multi-objective Evolutionary Algorithms and the Weighted Vertex Cover Problem

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## Abstract

Evolutionary multi-objective optimization for the classical vertex cover problem has been analysed in [12] in the context of parameterized complexity analysis. This paper extends the analysis to the weighted vertex cover problem in which integer weights are assigned to the vertices and the goal is to find a vertex cover of minimum weight. Using an alternative mutation operator introduced in [12], we provide a fixed parameter evolutionary algorithm with respect to  $OPT$ , the cost of an optimal solution for the problem. Moreover, we present a multi-objective evolutionary algorithm with standard mutation operator that keeps the population size in a polynomial order by means of a proper diversity mechanism; and therefore, manages to find a 2-approximation in expected polynomial time. We also introduce a population-based evolutionary algorithm which finds a  $(1 + \varepsilon)$ -approximation in expected time  $O(n \cdot 2^{\min\{n, 2(1-\varepsilon)OPT\}} + n^3)$ .

## 1 Introduction

The area of runtime analysis has provided many rigorous new insights into the working behaviour of bio-inspired computing methods such as evolutionary algorithms and ant colony optimization [1, 9, 16]. In recent years, the parameterized analysis of bio-inspired computing has gained additional interest [11, 12, 20, 21]. Here the runtime of bio-inspired computing is studied in dependence of the input size and additional parameters such as the solution size and/or other structural parameters of the given input.

One of the classical problems that has been studied extensively in the area of runtime analysis is the classical NP-hard vertex cover problem. Here, an undirected graph is given and the goal is to find a minimum set of vertices  $V'$  such that each edge has at least one endpoint in  $V'$ . Friedrich et al. [6] have shown that the single-objective evolutionary algorithm (1+1) EA can not achieve a constant approximation ratio in expected polynomial time. Furthermore, they have shown that a multi-objective approach using Global Simple Evolutionary Multi-objective Optimizer (Global SEMO) gives a factor  $O(\log n)$  approximation for the wider classes of set

cover problems in expected polynomial time. Further investigations regarding the approximation behaviour of evolutionary algorithms for the vertex cover problem have been carried out in [5, 17]. Edge-based representations in connection with different fitness functions have been investigated in [10, 18] according to their approximation behaviour in the static and dynamic setting. Kratsch and Neumann [12] have studied evolutionary algorithms and the vertex cover problem in the context of parameterized complexity [4]. They have shown that Global SEMO, with a problem specific mutation operator is a fixed parameter evolutionary algorithm for this problem (for details about fixed parameter evolutionary algorithms, please refer to [12]), and finds 2-approximations in expected polynomial time. Kratsch and Neumann [12] have also introduced an alternative mutation operator and have proved that Global SEMO using this mutation operator finds a  $(1 + \varepsilon)$ -approximation in expected time  $O(n^2 \log n + OPT \cdot n^2 + n \cdot 4^{(1-\varepsilon)OPT})$ . Jansen et al. [10] have shown that a 2-approximation can also be obtained by using an edge-based representation in the (1+1) EA combined with a fitness function formulation based on matchings.

In this paper, we consider the weighted vertex cover problem where integer weights on the vertices are given and the goal is to find a vertex cover of minimum weight. We extend the investigations carried out in [12] to the weighted minimum vertex cover problem. In [12], multi-objective models in combination with a simple multi-objective evolutionary algorithm called Global SEMO are investigated. The secondary objective that is studied there is the solution for the LP relaxation of the problem, which helps the evolutionary algorithm construct LP-based approximation solutions. One key argument for the results presented for the (unweighted) vertex cover problem is that the population size is always upper bounded by  $n + 1$ . This argument does not hold in the weighted case. Therefore, we study how a variant of Global SEMO using appropriate diversity mechanisms is able to deal with the weighted vertex cover problem.

The focus of this paper is on the expected time (number of fitness evaluations) of the algorithms to find good approximations of an optimal solution. The time complexity analysis is performed with respect to  $n$ ,  $W_{max}$ , and  $OPT$ , which denote the number of vertices, the maximum weight in the input graph, and the cost of the optimal solution respectively. We first study the expected time until Global SEMO with standard mutation operator has found a 2-approximation in dependence of  $n$  and  $OPT$ . Afterwards, we analyse the expected time that Global SEMO requires to find a solution with expected approximation ratio  $(1 + \varepsilon)$  for this problem when the algorithm uses an alternative mutation operator. Furthermore, this paper considers DEMO, a variant of Global SEMO, which incorporates  $\varepsilon$ -dominance [13] as a diversity mechanism. It is shown that DEMO finds a 2-approximation in expected polynomial time. Finally, a population-based approach is presented that obtains a solution that has approximation ratio  $(1 + \varepsilon)$  in expected time  $O(n \cdot 2^{\min\{n, 2(1-\varepsilon)OPT\}} + n^3)$ .

This paper extends the conference version [19] by giving complete proofs for a number of lemmata (Lemmata 4, 5, 11 and 12), that are not contained in the conference version. Furthermore, it analyses the expected time until Global SEMO with standard mutation operator has found a 2-approximation (Section 3.1), and provides a population-based approach that obtains a solution that has approximation ratio  $(1 + \varepsilon)$  in expected time  $O(n \cdot 2^{\min\{n, 2(1-\varepsilon)OPT\}} + n^3)$  (Section 5).

The outline of the paper is as follows. In Section 2, the problem definition is presented as well as the classical Global SEMO algorithm and DEMO algorithm. Runtime analysis for Global SEMO is presented in Section 3 with standard mutation operator investigated in Section 3.1 for finding a 2-approximation, and the alternative mutation operator analysed in Section 3.2 for finding a  $(1 + \varepsilon)$ -approximation. Section 4 includes the analysis that shows DEMO

can find 2–approximations of the optimum in expected polynomial time. The population-based algorithm is defined and investigated for finding a  $(1 + \varepsilon)$ –approximation in Section 5. At the end, in Section 6 we summarize and conclude.

## 2 Preliminaries

We consider the weighted vertex cover problem defined as follows. Given a graph  $G = (V, E)$  with vertex set  $V = \{v_1, \dots, v_n\}$  and edge set  $E = \{e_1, \dots, e_m\}$ , and a positive weight function  $w : V \rightarrow \mathbb{N}^+$  on the vertices, the goal is to find a subset of vertices,  $V_C \subseteq V$ , that covers all edges and has minimum weight, i.e.  $\forall e \in E, e \cap V_C \neq \emptyset$  and  $\sum_{v \in V_C} w(v)$  is minimized. We consider the standard node-based approach, i.e. the search space is  $\{0, 1\}^n$  and for a solution  $x = (x_1, \dots, x_n)$  the vertex  $v_i$  is chosen iff  $x_i = 1$ .

The weighted vertex cover problem has the following Integer Linear Programming (ILP) formulation.

$$\begin{aligned} \min \quad & \sum_{i=1}^n w(v_i) \cdot x_i \\ \text{st.} \quad & x_i + x_j \geq 1 \quad \forall [v_i, v_j] \in E \\ & x_i \in \{0, 1\} \quad \forall 1 \leq i \leq n \end{aligned}$$

By relaxing the constraint  $x_i \in \{0, 1\}$  to  $x_i \in [0, 1]$ , the linear program formulation of Fractional Weighted Vertex Cover is obtained. Hochbaum [7] has shown that we can find a 2-approximation using the LP result of the relaxed weighted vertex cover. This can be done by including any vertex  $v_i$  for which  $x_i \geq \frac{1}{2}$ .

We consider primarily multi-objective approaches for the weighted vertex cover problem. Given a multi-objective fitness function  $f = (f_1, \dots, f_d) : S \rightarrow \mathbb{R}^n$ , defined on the solution set  $S$ , where all  $d$  objectives should be minimized, we have  $f(x) \leq f(y)$  iff  $f_i(x) \leq f_i(y)$ ,  $1 \leq i \leq d$ . We say that  $x$  (weakly) dominates  $y$  iff  $f(x) \leq f(y)$ . Furthermore, we say that  $x$  (strongly) dominates  $y$  iff  $f(x) \leq f(y)$  and  $f(x) \neq f(y)$ . A solution  $x^*$  is *Pareto optimal* if there is no solution that can strongly dominate it. The set of Pareto optimal solutions is called *Pareto front*.

We now introduce the objectives used in our multi-objective evolutionary algorithm. Let  $G(x)$  be the graph obtained from  $G$  by removing all edges covered by the vertices chosen by  $x$ . Formally, we have  $G(x) = (V, E(x))$  where  $V(x) = V \setminus \{v_i \mid x_i = 1\}$  and  $E(x) = E \setminus \{e \mid e \cap (V \setminus V(x)) \neq \emptyset\}$  (note that to unify the search space, we keep  $G$  and  $G(x)$  the same vertex set). Kratsch and Neumann [12] investigated a multi-objective baseline algorithm called Global SEMO using the LP-value for  $G(x)$  as one of the fitness values for the (unweighted) minimum vertex cover problem.

Our goal is to expand the analysis on behaviour of multi-objective evolutionary algorithms to the Weighted Vertex Cover problem. In order to do this, we modify the fitness function that was used in Global SEMO in [12], to match the weighted version of the problem. We investigate the multi-objective fitness function  $f(x) = (Cost(x), LP(x))$ , where

- $Cost(x) = \sum_{i=1}^n w(v_i)x_i$  is the sum of weights of selected vertices
- $LP(x)$  is the value of an optimal solution of the LP for  $G(x)$ .

We analyse Global SEMO with this fitness function using the standard mutation operator flipping each bit with probability  $1/n$ . We also investigate Global SEMO using the alternative

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1 Choose  $x \in \{0, 1\}^n$  uniformly at random;
2 Determine  $f(x)$ ;
3  $P \leftarrow \{x\}$ ;
4 repeat
5   | Choose  $x \in P$  uniformly at random;
6   | Create  $x'$  by flipping each bit  $x_i$  of  $x$  with probability  $1/n$ ;
7   | Determine  $f(x')$ ;
8   | if  $\nexists y \in P \mid f(y) \leq f(x')$  then
9     |   |  $P \leftarrow \{x'\}$ ;
10    |   | delete all other solutions  $z \in P$  with  $f(x') \leq f(z)$  from  $P$ ;
11    | end
12 until termination condition satisfied;
```

**Algorithm 1:** Global SEMO

```

1 Choose  $b \in \{0, 1\}$  uniformly at random;
2 if  $(b = 1)$  then
3   | foreach  $i \in \{1, \dots, n\}$  do
4     |   | if  $\exists j \in \{1, \dots, n\} \mid \{v_i, v_j\} \in E(x)$  then
5       |   |   | Flip  $x_i$  with probability  $1/2$ ;
6       |   | else
7       |   |   | Flip  $x_i$  with probability  $1/n$ ;
8       |   | end
9     | end
10 else
11   | foreach  $i \in \{1, \dots, n\}$  do
12     |   | Flip  $x_i$  with probability  $1/n$ ;
13   | end
14 end
```

**Algorithm 2:** Alternative Mutation Operator

mutation operator introduced in [12] (see Algorithm 2). By this mutation operator, the vertices that are adjacent to uncovered edges are included with probability  $1/2$  in some steps.

In the fitness function used in Global SEMO, both  $Cost(x)$  and  $LP(x)$  can be exponential with respect to the input size; therefore, we need to deal with exponentially many solutions, even if we only keep the Pareto front. One approach for dealing with this problem is using the concept of  $\varepsilon$ -dominance [13]. The concept of  $\varepsilon$ -dominance has previously been proved to be useful for coping with exponentially large Pareto fronts in some problems [8, 15]. Having two objective vectors  $p = (p_1, \dots, p_m)$  and  $q = (q_1, \dots, q_m)$ ,  $p$   $\varepsilon$ -dominates  $q$ , denoted by  $p \preceq_\varepsilon q$ , if for all  $i \in \{1, \dots, m\}$  we have  $p_i \leq (1 + \varepsilon)q_i$ . Motivated by this approach, DEMO (Diversity Evolutionary Multi-objective Optimizer) has been investigated in [14, 15], which we present in Algorithm 3. In this approach, the objective space is partitioned into a polynomial number of boxes in which all solutions  $\varepsilon$ -dominate each other, and at most one solution from each box is kept in the population. Here we describe the concept of boxes and how we keep one solution for each box in detail, and in Section 4, we analyze DEMO.

```

1 Choose  $x \in \{0, 1\}^n$  uniformly at random;
2 Determine  $b(x)$ ;
3  $P \leftarrow \{x\}$ ;
4 repeat
5   Choose  $x \in P$  uniformly at random;
6   Create  $x'$  by flipping each bit  $x_i$  of  $x$  with probability  $1/n$ ;
7   Determine  $f(x')$  and  $b(x')$ ;
8   if  $\exists y \in P \mid (f(y) \leq f(x') \wedge f(y) \neq f(x')) \vee (b(y) = b(x') \wedge Cost(y) + 2 \cdot LP(y) \leq$ 
       $Cost(x') + 2 \cdot LP(x'))$  then
9     Go to 4;
10  else
11     $P \leftarrow \{x'\}$ ;
12    delete all other solutions  $z \in P$  where  $f(x') \leq f(z) \vee b(z) = b(x')$  from  $P$ ;
13  end
14 until termination condition satisfied;
```

**Algorithm 3:** DEMO

To implement the concept of  $\varepsilon$ -dominance in DEMO, we use the parameter  $\delta = \frac{1}{2n}$  and define the boxing function  $b : \{0, 1\}^n \rightarrow \mathbb{N}^2$  as:

$$\begin{aligned}
b_1(x) &= \lceil \log_{1+\delta}(1 + Cost(x)) \rceil, \\
b_2(x) &= \lceil \log_{1+\delta}(1 + LP(x)) \rceil,
\end{aligned}$$

The functions  $b_1$  and  $b_2$  partition the objective space into horizontal and vertical stripes, which we name *rows* and *columns*, and the whole boxing function partitions the objective space into boxes. A box can be denoted by  $B = (a, b)$ , where  $a$  and  $b$  are values of  $b_1$  and  $b_2$  for the solutions in that box, respectively.

Note that two boxes  $B = (a, b)$  and  $B' = (a', b')$  with  $a = a'$  and  $b < b'$  (or  $a < a'$  and  $b = b'$ ) can include search points that do not dominate each other; therefore, we may keep solutions from different boxes with same values of  $b_1$  or  $b_2$ . But if  $a < a'$  and  $b < b'$ , then all search points in  $B$  dominate all search points in  $B'$ . Hence, we define dominance among boxes as: box  $B = (a, b)$  dominates box  $B' = (a', b')$ , denoted by  $B < B'$ , if  $a < a'$  and  $b < b'$ .

In DEMO only one non-dominated solution can be kept in the population for each box based on a predefined criteria. In our setting, among two solutions  $x$  and  $y$  from one box,  $y$  is kept in  $P$  and  $x$  is discarded if  $Cost(y) + 2 \cdot LP(y) \leq Cost(x) + 2 \cdot LP(x)$ . The reason behind this particular setting is that we aim to work on solutions  $x$  under the constraint that  $Cost(x) + 2 \cdot LP(x) \leq 2 \cdot OPT$ , because by only adding vertices to these solutions, it is possible to obtain 2-approximate complete vertex covers.

Analysing the runtime of our evolutionary algorithms, we are interested in the expected number of rounds of the repeat loop until a solution of desired quality has been obtained. We call this the expected time until the considered algorithm has achieved its desired goal.

### 3 Analysis of Global SEMO

In this section we analyse the expected time of Global SEMO to find good approximations for the weighted vertex cover problem in dependence of the input size and OPT. Before we present

our analysis for Global SEMO, we state some basic properties of the solutions in our multi-objective model. The following theorem shown by Balinski [2] states that all basic feasible solutions of the LP relaxation of the weighted vertex cover, which are the extremal points or the corner solutions of the polyhedron that forms the feasible space, are half-integral.

**Theorem 1.** *Each basic feasible solution  $x$  of the LP relaxation of the weighted vertex cover is half-integral, i.e.,  $x \in \{0, 1/2, 1\}^n$ . [2]*

As a result, there always exists a half integral optimal LP solution for a vertex cover problem. In several parts of this paper, we make use of this result. We establish the following two lemmata which we will use later on in the analysis of our algorithms.

**Lemma 2.** *For any  $x \in \{0, 1\}^n$ ,  $LP(x) \leq LP(0^n) \leq OPT$ .*

*Proof.* Let  $y$  be the LP solution of  $LP(0^n)$ . The solution  $0^n$  contains no vertices; therefore,  $y$  is the optimal fractional vertex cover for all edges of the input graph. Thus, for any solution  $x$ ,  $y$  is a (possibly non-optimal) fractional cover for  $G(x)$ ; therefore,  $LP(x) \leq LP(0^n)$ . Moreover, we have  $LP(0^n) \leq OPT$  as  $LP(0^n)$  is the optimal value of the LP relaxation. ■

**Lemma 3.** *Let  $x = \{x_1, \dots, x_n\}, x_i \in \{0, 1\}$  be a solution and  $y = \{y_1, \dots, y_n\}, y_i \in [0, 1]$  be a fractional solution for  $G(x)$ . If there is a vertex  $v_i$  where  $y_i \geq \frac{1}{2}$ , mutating  $x_i$  from 0 to 1 results in a solution  $x'$  for which  $LP(x') \leq LP(x) - y_i \cdot w(v_i) \leq LP(x) - \frac{1}{2}w(v_i)$ .*

*Proof.* The graph  $G(x')$  is the same as  $G(x)$  excluding the edges connected to  $v_i$ . Therefore, the solution  $y' = \{y_1, \dots, y_{i-1}, 0, y_{i+1}, y_n\}$  is a fractional vertex cover for  $G(x')$  and has a cost of  $LP(x) - y_i w(v_i)$ . The cost of the optimal fractional vertex cover of  $G(x')$  is at most as great as the cost of  $y'$ ; thus  $LP(x') \leq LP(x) - y_i w(v_i) \leq LP(x) - \frac{1}{2}w(v_i)$ . ■

### 3.1 2-Approximation

We now analyse the runtime behaviour of Global SEMO (Algorithm 1) with the standard mutation operator, in dependence of  $OPT$ . We start by giving an upper bound on the population size of Global SEMO.

**Lemma 4.** *The population size of Algorithm 1 is upper bounded by  $2 \cdot OPT + 1$ .*

*Proof.* For any solution  $x$  there exists an optimal fractional vertex cover which is half-integral (Theorem 1). Moreover, we are assuming that all the weights are integer values. Therefore,  $LP(x)$  can only take  $2LP(0^n) + 1$  different values, because  $LP(0^n)$  is an upper bound on  $LP(x)$  (Lemma 2). For each value of  $LP$ , only one solution is in  $P$ , because Algorithm 1 keeps non-dominated solutions only. Therefore, the population size of this algorithm is upper bounded by  $2 \cdot LP(0^n) + 1$  which is at most  $2 \cdot OPT + 1$  due to Lemma 2. ■

For our analysis, we first consider the expected time of Global SEMO to reach a population which contains the empty set of vertices. Once included, such a solution will never be removed from the population as it is minimal with respect to the cost function.

**Lemma 5.** *The search point  $0^n$  is included in the population in expected time of  $O(OPT \cdot n(\log W_{max} + \log n))$ .*

*Proof.* From Lemma 4 we know that the population contains at most  $2 \cdot OPT + 1$  solutions. Therefore, at each step, there is a probability of  $\frac{1}{2 \cdot OPT + 1}$  that the solution  $x_{min}$  is selected where  $Cost(x_{min}) = \min_{x \in P} Cost(x)$ .

If  $Cost(x_{min}) > 0$ , there must be  $k \geq 1$  vertices such as  $v_i$  in  $x_{min}$  where  $x_i = 1$ . Let  $\Delta^t$  be the improvement that happens on the minimum cost in  $P$  at step  $t$ . If all the 1-bits in solution  $x_{min}$  flip to zero, at the same step or different steps, a solution  $0^n$  will be obtained with  $Cost(0^n) = 0$ , which implies that the expected improvement that flipping a randomly chosen 1-bit makes is  $\Delta^t = \frac{Cost(x_{min})}{k}$  at each step  $t$ . Note that flipping 1-bits always improves the minimum cost and the new solution is added to the population. Moreover, flipping any 0-bits does not improve the minimum cost in the population and  $x_{min}$  is not replaced with the new solution in that case.

At each step, with probability at least  $\frac{1}{e}$  only one bit flips. With probability  $\frac{k}{n}$ , the flipping bit is a 1-bit, and makes an expected improvement of  $\Delta^t = \frac{Cost(x_{min})}{k}$ , and with probability  $1 - \frac{k}{n}$ , a 0-bit is flipped with  $\Delta^t = 0$ . We can conclude that the expected improvement of minimum cost, when only one bit of  $x_{min}$  flips, is at least

$$\frac{k}{n} \cdot \frac{Cost(x_{min})}{k} = \frac{Cost(x_{min})}{n}$$

Moreover, the algorithm selects  $x_{min}$  and flips only one bit with probability at least  $\frac{1}{(2 \cdot OPT + 1) \cdot e}$ ; therefore, the expected improvement of minimum cost is bounded by

$$E[\Delta^t | x_{min}] \geq \frac{Cost(x_{min})}{(2 \cdot OPT + 1) \cdot e \cdot n}$$

The maximum value that  $Cost(x_{min})$  can take is bounded by  $W_{max} \cdot n$ , and for any solution  $x \neq 0^n$ , the minimum value of  $Cost(x)$  is at least 1. Using Multiplicative Drift Analysis [3] with  $s_0 \leq W_{max} \cdot n$  and  $s_{min} \geq 1$ , we can conclude that in expected time  $O(OPT \cdot n(\log W_{max} + \log n))$  solution  $0^n$  is included in the population. ■

We now show that Global SEMO is able to achieve a 2-approximation efficiently as long as OPT is small.

**Theorem 6.** *The expected number of iterations of Global SEMO until the population  $P$  contains a 2-approximation is  $O(OPT \cdot n(\log W_{max} + \log n))$ .*

*Proof.* Let  $x$  be a solution that minimizes  $LP(x)$  under the constraint that  $Cost(x) + 2 \cdot LP(x) \leq 2 \cdot OPT$ . Note that this constraint holds for solution  $0^n$  since  $LP(0^n) \leq OPT$ , and according to Lemma 5, solution  $0^n$  exists in the population in expected time of  $O(OPT \cdot n(\log W_{max} + \log n))$ .

If  $LP(x) = 0$ , then all edges are covered and  $x$  is a 2-approximate vertex cover, because we have  $Cost(x) + 2 \cdot LP(x) \leq 2 \cdot OPT$  as the constraint. Otherwise, some edges are uncovered and any LP solution of  $G(x)$  assigns at least  $\frac{1}{2}$  to at least one vertex of any uncovered edge. Let  $y = \{y_1, \dots, y_n\}$  be a basic LP solution for  $G(x)$ . According to Theorem 1,  $y$  is a half-integral solution.

Let  $\Delta^t$  be the improvement that happens on the minimum  $LP$  value among solutions that fulfil the constraint at time step  $t$ . Also, let  $k$  be the number of vertices that are assigned at least  $\frac{1}{2}$  by  $y$ . Flipping only one of these vertices by the algorithm happens with probability at least  $\frac{k}{e \cdot n}$ . According to Lemma 3, flipping one of these vertices,  $v_i$ , results in a solution  $x'$  with  $LP(x') \leq LP(x) - \frac{1}{2}w(v_i)$ . Observe that the constraint of  $Cost(x') + 2 \cdot LP(x') \leq 2 \cdot OPT$  holds for solution  $x'$ . Therefore,  $\Delta^t \geq y_i \cdot w(v_i)$ , which is on expectation at least  $\frac{LP(x)}{k}$  due to definition

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of  $LP(x)$ . Moreover, at each step, the probability that  $x$  is selected and only one of the  $k$  bits defined above flips is at least  $\frac{k}{(2 \cdot OPT + 1) \cdot e \cdot n}$ . As a result we have:

$$E[\Delta^t | x] \geq \frac{k}{(2 \cdot OPT + 1) \cdot e \cdot n} \cdot \frac{LP(x)}{k} = \frac{LP(x)}{en(2 \cdot OPT + 1)}$$

According to Lemma 2 for any solution  $x$ , we have  $LP(x) \leq OPT$ . We also know that for any solution  $x$  which is not a complete cover,  $LP(x) \geq 1$ , because the weights are positive integers. Using the method of Multiplicative Drift Analysis [3] with  $s_0 \leq OPT$  and  $s_{min} \geq 1$ , in expected time of  $O(OPT \cdot n \log OPT)$  a solution  $y$  with  $LP(y) = 0$  and  $Cost(y) + 2LP(y) \leq 2OPT$  is obtained which is a 2-approximate vertex cover. Overall, since we have  $OPT \leq W_{max} \cdot n$ , the expected time of finding this solution is  $O(OPT \cdot n(\log W_{max} + \log n))$ . ■

### 3.2 Improved Approximations by Alternative Mutation

In this section, we analyse the expected time of Global SEMO with alternative mutation operator to find a  $(1+\epsilon)$ -approximation.

**Lemma 7.** *A solution  $x$  fulfilling the two properties*

1.  $LP(x) = LP(0^n) - Cost(x)$  and
2. *there is an optimal solution of the LP for  $G(x)$  which assigns 1/2 to each non-isolated vertex of  $G(x)$*

*is included in the population of Global SEMO in expected time  $O(OPT \cdot n(\log W_{max} + \log n + OPT))$ .*

*Proof.* As the standard mutation occurs with probability 1/2 in the alternative mutation operator, the search point  $0^n$  which satisfies property 1 is included in the population in expected time of  $O(OPT \cdot n(\log W_{max} + \log n))$  using the argument presented in the proof of Lemma 5. Let  $P' \subseteq P$  be a set of solutions such that for each solution  $x \in P'$ ,  $LP(x) + Cost(x) = LP(0^n)$ . Let  $x_{min} \in P'$  be a solution such that  $LP(x_{min}) = \min_{x \in P'} LP(x)$ .

If the optimal fractional vertex cover for  $G(x_{min})$  assigns 1/2 to each non-isolated vertex of  $G(x_{min})$ , then the conditions of the lemma hold. Otherwise, it assigns 1 to some non-isolated vertex, say  $v$ . The probability that the algorithm selects  $x_{min}$  and flips the bit corresponding to  $v$ , is  $\Omega(\frac{1}{OPT \cdot n})$ , because the population size is  $O(OPT)$  (Lemma 4). Let  $x_{new}$  be the new solution. We have  $Cost(x_{new}) = Cost(x_{min}) + w(v)$ , and by Lemma 3,  $LP(x_{new}) \leq LP(x_{min}) - w(v)$ . This implies that  $LP(x_{new}) + Cost(x_{new}) = LP(0^n)$ ; hence,  $x_{new}$  is a Pareto Optimal solution and is added to the population  $P$ .

Since  $LP(x_{min}) \leq OPT$  (Lemma 2) and the weights are at least 1, assuming that we already have the solution  $0^n$  in the population, by means of the method of fitness based partitions, we find the expected time of finding a solution that fulfils the properties given above as  $O(OPT^2 \cdot n)$ . Since the search point  $0^n$  is included in expected time  $O(OPT \cdot n(\log W_{max} + \log n))$ , the expected time that a solution fulfilling the properties given above is included in  $P$  is  $O(OPT \cdot n(\log W_{max} + \log n + OPT))$ . ■

We now present the main approximation result for Global SEMO using the alternative mutation operator. The general idea of the analysis given in the following theorem is to partition

the non-isolated vertices in  $G(x)$  into four subsets:  $S_1$ ,  $S_2$ ,  $T_1$ , and  $T_2$ , where  $x$  denotes a solution satisfying the two properties given in Lemma 7. The precise definition of these four subsets are given in the proof of the theorem. For a new vertex cover  $x'$  obtained by the alternative mutation operator on  $x$ , the analysis only considers the probability that all vertices of  $S_1$  are chosen and no vertex of  $T_1$  is chosen in the new solution  $x'$ . The quality of  $x'$  highly depends on this property (including all vertices of  $S_1$  and no vertices of  $T_1$ ), but it also depends on the vertices chosen from  $S_2$  and  $T_2$ , where the vertices in  $S_2$  and  $T_2$  are chosen randomly with probability  $1/2$  by the alternative mutation operator. Thus the analysis find the expected time until the event that a solution  $x'$  with the defined property is found, and the *expected* ratio of that solution is considered, based on the expectation that half of the vertices in  $S_2$  and half of the vertices in  $T_2$  are chosen by  $x'$ . In the following, we first present the formal definition of sets  $S_1$  and  $T_1$  and the mentioned property, and then we state the main theorem of this section.

**Definition 8.** *Let  $x$  be a solution that satisfies the two properties given in Lemma 7. Also, let  $X$  be the set containing all non-isolated vertices in graph  $G(x)$ . Moreover, let  $S \subseteq X$  be a vertex cover of  $G(x)$  with the minimum weight over all vertex covers of  $G(x)$ , and  $T$  be the set containing all non-isolated vertices in  $X \setminus S$ . For a set of vertices,  $X'$ , we define  $Cost(X') = \sum_{v \in X'} w(v)$ . Let  $OPT' = OPT - Cost(x)$ . Let  $s_1, \dots, s_{|S|}$  be a numbering of the vertices in  $S$  such that  $w(s_i) \leq w(s_{i+1})$ , for all  $1 \leq i \leq |S| - 1$ . And let  $t_1, \dots, t_{|T|}$  be a numbering of the vertices in  $T$  such that  $w(t_i) \geq w(t_{i+1})$ , for all  $1 \leq i \leq |T| - 1$ . We define*

- $S_1 = \{s_1, s_2, \dots, s_\rho\}$ , where  $\rho = \min\{|S|, \lceil (1 - \varepsilon) \cdot OPT' \rceil\}$
- $T_1 = \{t_1, t_2, \dots, t_\eta\}$ , where  $\eta = \min\{|T|, \lceil (1 - \varepsilon) \cdot OPT' \rceil\}$

**Property 9 (High-Quality solutions).** *We say that a solution  $x$  has the property of a High-Quality solution if all vertices of  $S_1$  are chosen and no vertex of  $T_1$  is chosen in  $x$ .*

**Theorem 10.** *The expected time until Global SEMO has obtained a solution with Property 9 (High-Quality solution) is  $O(OPT \cdot 2^{\min\{n, 2(1-\varepsilon)OPT\}} + OPT \cdot n(\log W_{max} + \log n + OPT))$ . Moreover, the obtained solution has expected approximation ratio of  $(1 + \varepsilon)$ .*

*Proof.* By Lemma 7, a solution  $x$  that satisfies the two properties given in Lemma 7 is included in the population in expected time of  $O(OPT \cdot n(\log W_{max} + \log n + OPT))$ . Let  $X$ ,  $S$ ,  $T$ ,  $\rho$ ,  $\eta$ , and  $Cost(X')$  for a vertex set  $X'$  be as defined in Definition 8. Due to property 2 of Lemma 7,  $\frac{1}{2}Cost(S) + \frac{1}{2}Cost(T) = LP(x) \leq Cost(S)$ ; therefore,  $Cost(T) \leq Cost(S)$ . Also, let  $OPT'$  be as defined in Definition 8. Observe that  $OPT' = Cost(S)$ , because  $S$  is the minimum vertex cover of  $G(x)$ .

With probability  $\Omega(\frac{1}{OPT})$ , the algorithm Global SEMO selects the solution  $x$ , and sets  $b = 1$  in the Alternative Mutation Operator. With  $b = 1$ , the probability that the bits corresponding to all vertices of  $S_1$  are flipped, is  $\Omega((\frac{1}{2})^\rho)$ , and the probability that none of the bits corresponding to the vertices of  $T_1$  are flipped is  $\Omega((\frac{1}{2})^\eta)$ . Also, the bits corresponding to the isolated vertices of  $G(x)$  are flipped with probability  $\frac{1}{n}$  by the Alternative Mutation Operator; hence, the probability that none of them flips is  $\Omega(1)$ . As a result, with probability  $\Omega(\frac{1}{OPT} \cdot (\frac{1}{2})^{\rho+\eta})$ , solution  $x$  is selected, the vertices of  $S_1$  are included, and the vertices of  $T_1$  and isolated vertices are not included in the new solution  $x'$ . Due to the definition of Property 9,  $x'$  has the property of a *High-Quality solution*. Since  $\rho + \eta \leq 2\lceil (1 - \varepsilon) \cdot OPT' \rceil \leq 2\lceil (1 - \varepsilon) \cdot OPT \rceil$ , and also  $\rho + \eta \leq n$ ; the expected time until solution  $x'$  is found after reaching solution  $x$ , is  $O(OPT \cdot 2^{\min\{n, 2(1-\varepsilon)OPT\}})$ .

Now we show that the second statement of the theorem holds. Note that the bits corresponding to vertices of  $S_2 = S \setminus S_1$  and  $T_2 = T \setminus T_1$ , are arbitrarily flipped in solution  $x'$  with

probability  $1/2$  by the Alternative Mutation Operator. Here we show that for the expected cost and the LP value of  $x'$ , the following constraint holds:  $E[Cost(x')] + 2 \cdot LP(x') \leq (1 + \varepsilon) \cdot OPT$ .

Let  $S' \subseteq S$  and  $T' \subseteq T$  denote the subset of vertices of  $S$  and  $T$  that are actually included in the new solution  $x'$  respectively. In the following, we show that for the expected values of  $Cost(S')$  and  $Cost(T')$ , we have:

$$E[Cost(S')] \geq (1 - \varepsilon) \cdot OPT' + E[Cost(T')] \quad (1)$$

Since the bits corresponding to the vertices of  $S_2$  and  $T_2$  are flipped with probability  $1/2$ , for the expected values of  $Cost(S')$  and  $Cost(T')$  we have:

$$\begin{aligned} E[Cost(S')] &= Cost(S_1) + \frac{Cost(S_2)}{2} \\ &= Cost(S_1) + \frac{Cost(S) - Cost(S_1)}{2} \\ &= 1/2 Cost(S) + 1/2 Cost(S_1) \end{aligned}$$

and

$$E[Cost(T')] = 1/2 Cost(T_2)$$

If  $\rho = |S|$ , then  $S_1 = S$  and  $Cost(S_1) = Cost(S) = OPT'$ . If  $\rho = \lceil (1 - \varepsilon) \cdot OPT' \rceil$ , we have  $Cost(S_1) \geq (1 - \varepsilon) \cdot OPT'$ , since each vertex has a weight of at least 1. Using  $Cost(S) = OPT'$  and the inequality above, we have

$$E[Cost(S')] \geq (1 - \varepsilon) \cdot OPT' + \frac{\varepsilon \cdot OPT'}{2}$$

We divide the analysis into two cases based on the relation between  $\eta$  and  $|T|$ .

Case (I).  $\eta = |T|$ . Then  $T_2 = T' = \emptyset$ . Thus,  $E[Cost(T')] = 0$  and Inequality (1) holds true.

Case (II).  $\eta = \lceil (1 - \varepsilon) \cdot OPT' \rceil < |T|$ . Since  $w(t_i) \geq w(t_{i+1})$  for  $1 \leq i \leq |T| - 1$  and  $Cost(T) \leq Cost(S) = OPT'$ , we have

$$\begin{aligned} Cost(T_2) &\leq \frac{|T| - \eta}{|T|} Cost(T) \\ &\leq \frac{OPT' - \lceil (1 - \varepsilon) \cdot OPT' \rceil}{OPT'} Cost(T) \\ &\leq \frac{OPT' - (1 - \varepsilon) \cdot OPT'}{OPT'} Cost(T) \\ &\leq \varepsilon \cdot Cost(S) = \varepsilon \cdot OPT' \end{aligned}$$

Thus for the expected value of  $Cost(T')$ , we have

$$E[Cost(T')] = \frac{1}{2} Cost(T_2) \leq \frac{\varepsilon \cdot OPT'}{2}$$

Summarizing above analysis, we can get that the Inequality 1 holds. In the following, using Inequality (1), we prove that, on expectation, the new solution  $x'$  satisfies the inequality  $Cost(x') + 2 \cdot LP(x') \leq (1 + \varepsilon) \cdot OPT$ .

$$E[Cost(x')] + 2 \cdot LP(x')$$

$$\begin{aligned}
&= \text{Cost}(x) + E[\text{Cost}(S')] + E[\text{Cost}(T')] + 2 \cdot LP(x') \\
&\leq \text{Cost}(x) + E[\text{Cost}(S')] + E[\text{Cost}(S')] - (1 - \varepsilon) \cdot OPT' + 2 \cdot LP(x') \\
&\leq \text{Cost}(x) + 2E[\text{Cost}(S')] - (1 - \varepsilon) \cdot OPT' + 2 \cdot (OPT' - E[\text{Cost}(S')]) \\
&= \text{Cost}(x) + (1 + \varepsilon) \cdot OPT' = \text{Cost}(x) + (1 + \varepsilon) \cdot (OPT - \text{Cost}(x)) \\
&\leq (1 + \varepsilon) \cdot OPT.
\end{aligned}$$

The third inequality holds because the set  $S_1$  chosen by  $x$  is a subset of the optimal solution for  $G(x)$ .

Now we analyze whether the new solution  $x'$  could be included in the population  $P$ . If  $x'$  could not be included in  $P$ , then there is a solution  $x''$  dominating  $x$ , i.e.,  $LP(x'') \leq LP(x')$  and  $\text{Cost}(x'') \leq \text{Cost}(x')$ . This implies  $\text{Cost}(x'') + 2 \cdot LP(x'') < \text{Cost}(x') + 2 \cdot LP(x') \leq (1 + \varepsilon) \cdot OPT$ . Therefore, after having a solution that fulfils the properties of Lemma 7 in  $P$ , in expected time  $O(OPT \cdot 2^{\min\{n, 2(1-\varepsilon)OPT\}})$ , the population would contain a solution  $y$  such that  $\text{Cost}(y) + 2 \cdot LP(y) \leq (1 + \varepsilon) \cdot OPT$ .

Let  $P'$  contain all solutions  $x \in P$  such that  $\text{Cost}(x) + 2 \cdot LP(x) \leq (1 + \varepsilon) \cdot OPT$ , and let  $x_{min}$  be the one that minimizes  $LP$ . With similar proof as we saw in Theorem 6 it is possible to show that at each step, on expectation  $LP(x_{min})$  improves by  $\frac{LP(x)}{en(2 \cdot OPT + 1)}$ . Using Multiplicative Drift Analysis, we get the expected time  $O(OPT \cdot n \log OPT)$  to find a solution  $y$  for which  $LP(y) = 0$  and  $\text{Cost}(y) + 2 \cdot LP(y) \leq (1 + \varepsilon) \cdot OPT$ .

Overall, the expected number of iterations of Global SEMO with alternative mutation operator, for getting a weighted vertex cover with expected approximation ratio  $(1 + \varepsilon)$ , is bounded by  $O(OPT \cdot 2^{\min\{n, 2(1-\varepsilon)OPT\}} + OPT \cdot n(\log W_{max} + \log n + OPT))$ . ■

## 4 Analysis of DEMO

Due to Lemma 4, with Global SEMO, the population size is upper bounded by  $O(OPT)$ , which can be exponential in terms of the input size. In this section, we analyse the other evolutionary algorithm, DEMO (Algorithm 3), that uses some diversity handling mechanisms for dealing with exponentially large population sizes. The following lemmata are used in the proof of Theorem 14.

**Lemma 11.** *Let  $W_{max}$  be the maximum weight assigned to a vertex. The population size of DEMO is upper bounded by  $O(n \cdot (\log n + \log W_{max}))$ .*

*Proof.* The values that can be taken by  $b_1$  are integer values between 0 and  $\lceil \log_{1+\delta}(1 + \text{Cost}(1^n)) \rceil$  and the values that can be taken by  $b_2$  are integer values between 0 and  $\lceil \log_{1+\delta}(1 + LP(0^n)) \rceil$  (Lemma 2). Since  $n \cdot W_{max}$  is an upper bound for both  $\text{Cost}(1^n)$  and  $LP(0^n)$ , the number of rows and also the number of columns are bounded by

$$\begin{aligned}
k &= (1 + \lceil \log_{1+\delta}(1 + n \cdot W_{max}) \rceil) \\
&\leq \left(1 + \lceil \frac{\log(1 + n \cdot W_{max})}{\log(1 + \delta)} \rceil\right) \\
&= O(n \cdot (\log n + \log W_{max}))
\end{aligned}$$

The last equality holds because  $\delta = \frac{1}{2n}$ .

We here show that the size of the population is  $P_{size} \leq 2k - 1$ . Since the dominated solutions according to  $f$  are discarded by the algorithm, none of the solutions in  $P$  can be located in a box that is dominated by another box that contains a solution in  $P$ . Moreover, at most one solution from each box is kept in the population; therefore,  $P_{size}$  is at most the maximum number of boxes where none of them dominates another.

Let  $k_1$  be the number of boxes that contain a solution of  $P$  in the first column. Let  $r_1$  be the smallest row number among these boxes. Observe that  $r_1 \leq k - k_1 + 1$  and the equality holds when the boxes are from rows  $k$  down to  $k - k_1 + 1$ . Any box in the second column with a row number of  $r_1 + 1$  or above is dominated by the box of the previous column and row  $r_1$ . Therefore, the maximum row number for a box in the second column, that is not dominated, is  $r_1 \leq k - k_1 + 1$ . With generalizing the idea, the maximum row number for a box in the column  $i$ , that is not dominated, is  $r_{i-1} \leq k - k_1 - \dots - k_{i-1} + i - 1$ , where for  $1 \leq j \leq k$ ,  $k_j$  is the number of boxes that contain a solution of  $P$  in column  $j$ .

The last column has  $k_k \leq r_{k-1}$  boxes which gives us:

$$k_k \leq r_{k-1} \leq k - k_1 - \dots - k_{k-1} + k - 1$$

This implies that

$$k_1 + \dots + k_k \leq r_{k-1} \leq 2k - 1$$

which completes the proof. ■

**Lemma 12.** *The search point  $x_z = 0^n$  is included in the population in expected time of  $O(n^3(\log n + \log W_{max})^2)$ .*

*Proof.* From Lemma 11 we know that the population contains  $P_{size} = O(n \cdot (\log n + \log W_{max}))$  solutions. Therefore, at each step, there is a probability of at least  $\frac{1}{P_{size}}$  that the solution  $x_{min}$  is selected where  $b_1(x_{min}) = \min_{x \in P} b_1(x)$ .

If  $b_1(x_{min}) = 0$ , we have  $Cost(x_{min}) = 0$ , which means  $x_{min} = 0^n$  since the weights are greater than 0.

If  $b_1(x_{min}) \neq 0$ , there must be at least one vertex  $v_i$  in  $x_{min}$  where  $x_i = 1$ . Consider  $v_j$  the vertex that maximizes  $w(v_i)$  among vertices  $v_i$  where  $x_i = 1$ . If  $Cost(x) = C$ , then  $w(v_j) \geq \frac{C}{n}$ , because  $n$  is an upper bound on the number of vertices selected by  $x_{min}$ . As a result, removing vertex  $x_j$  from solution  $x_{min}$  results in a solution  $x'$  for which  $Cost(x') \leq C \cdot (1 - \frac{1}{n})$ . Using this value of  $Cost(x')$ , we have

$$\begin{aligned} (1 + \delta)(1 + Cost(x')) &\leq 1 + \delta + C(1 - \frac{1}{n})(1 + \delta) \\ &\leq 1 + \delta + C + C(\delta - \frac{1}{n} - \frac{\delta}{n}) \\ &\leq 1 + C\delta + C + C(\delta - \frac{1}{n} - \frac{\delta}{n}) \\ &\leq 1 + C + C(2\delta - \frac{1}{n} - \frac{\delta}{n}) \\ &\leq 1 + C \end{aligned}$$

The third inequality above holds because  $C \geq 1$  and the last one holds because  $\delta = \frac{1}{2n}$ . From  $(1 + \delta)(1 + Cost(x')) \leq 1 + C$  we can observe that

$$1 + \log_{1+\delta}(1 + Cost(x')) \leq \log_{1+\delta}(1 + C)$$

which implies  $b_1(x') \leq b_1(x) - 1$ . Note that  $x'$  is obtained by performing a 1-bit flip on  $x$  and is done at each step with a probability of at least

$$\begin{aligned} & \frac{1}{P_{size}} \cdot \frac{1}{n} \cdot \left(1 - \frac{1}{n}\right)^{n-1} \\ &= \Omega\left(\frac{1}{n(\log n + \log W_{max})} \cdot \frac{1}{n}\right) \end{aligned}$$

Therefore, in expected time of at most  $O(n^2(\log n + \log W_{max}))$  the new solution,  $x'$  is obtained which is accepted by the algorithm because it is placed in a box with a smaller value of  $b_1$  than all solutions in  $P$  and hence not dominated. There are  $O(n(\log n + \log W_{max}))$  different values for  $b_1$ ; therefore, the solution  $x_z = 0^n$  with  $b_1(x_z) = 0$  is found in expected time of at most  $O(n^3(\log n + \log W_{max})^2)$ . ■

**Lemma 13.** *Let  $x \in P$  be a search point such that  $Cost(x) + 2 \cdot LP(x) \leq 2 \cdot OPT$  and  $b_2(x) > 0$ . There exists a 1-bit flip leading to a search point  $x'$  with  $Cost(x') + 2 \cdot LP(x') \leq 2 \cdot OPT$  and  $b_2(x') < b_2(x)$ .*

*Proof.* Let  $y = \{y_1 \dots y_n\}$  be a basic half integral LP solution for  $G(x)$ . Since  $b_2(x) = LP(x) \neq 0$ , there must be at least one uncovered edge; hence, at least one vertex  $v_i$  has a  $y_i \geq \frac{1}{2}$  in LP solution  $y$ . Consider  $v_j$  the vertex that maximizes  $y_i w(v_i)$  among vertices  $v_i$ ,  $1 \leq i \leq n$ . Also, let  $x'$  be a solution obtained by adding  $v_j$  to  $x$ . Since solutions  $x$  and  $x'$  are only different in one vertex,  $v_j$ , we have  $Cost(x') = Cost(x) + w(v_j)$ . Moreover, according to Lemma 3,  $LP(x') \leq LP(x) - \frac{1}{2} \cdot w(v_j)$ . Therefore,

$$\begin{aligned} Cost(x') + 2 \cdot LP(x') &\leq Cost(x) + w(v_j) + 2 \left( LP(x) - \frac{w(v_j)}{2} \right) \\ &\leq Cost(x) + 2 \cdot LP(x) \leq 2 \cdot OPT \end{aligned}$$

which means solution  $x'$  fulfils the mentioned constraint. If  $LP(x) = W$ , then  $y_j w(v_j) \geq \frac{W}{n}$ , because  $n$  is an upper bound on the number of vertices selected by the LP solution. As a result, using Lemma 3, we get  $LP(x') \leq W \cdot \left(1 - \frac{1}{n}\right)$ . Therefore, with similar analysis as Lemma 12 we get:

$$\begin{aligned} (1 + \delta)(1 + LP(x')) &\leq 1 + \delta + W \left(1 - \frac{1}{n}\right)(1 + \delta) \\ &\leq 1 + W \end{aligned}$$

This inequality implies

$$1 + \log_{1+\delta}(1 + LP(x')) \leq \log_{1+\delta}(1 + W)$$

As a result,  $b_2(x') < b_2(x)$  holds for  $x'$ , which is obtained by performing a 1-bit flip on  $x$ , and the lemma is proved. ■

**Theorem 14.** *The expected time until DEMO constructs a 2-approximate vertex cover is  $O(n^3 \cdot (\log n + \log W_{max})^2)$ .*

```

1 Choose  $x \in \{0, 1\}^n$  uniformly at random;
2  $P \leftarrow \{x\}$ ;
3 repeat
4   Choose  $x \in P$  uniformly at random;
5   Create  $x'$  by using Alternative Mutation Operator;
6    $P \leftarrow \{x'\}$ ;
7   Let  $P'$  be a set containing all solutions  $y \in P$  where  $|y|_1 = |x'|_1$ ;
8   Find solutions  $y_{min_1}$  and  $y_{min_2}$  from  $P'$  such that  $y_{min_1}$  minimizes  $Cost(z) + LP(z)$ ,
   and  $y_{min_2}$  minimizes  $Cost(z) + 2 \cdot LP(z)$  among solutions  $z \in P'$ ;
9    $P = P \setminus P'$ ;
10   $P \leftarrow \{y_{min_1}, y_{min_2}\}$ ;
11 until termination condition satisfied;

```

**Algorithm 4:** Diverse Population-Based EA

*Proof.* Consider solution  $x \in P$  that minimizes  $b_2(x)$  under the constraint that  $Cost(x) + 2 \cdot LP(x) \leq 2 \cdot OPT$ . Note that  $0^n$  fulfils this constraint and according to Lemma 12, the solution  $0^n$  will be included in  $P$  in time  $O(n^3(\log n + \log W_{max})^2)$ .

If  $b_2(x) = 0$  then  $x$  covers all edges and by selection of  $x$  we have  $Cost(x) \leq 2 \cdot OPT$ , which means that  $x$  is a 2-approximation.

In case  $b_2(x) \neq 0$ , according to Lemma 13 there is a one-bit flip on  $x$  that results in a new solution  $x'$  for which  $b_2(x') < b_2(x)$ , while the mentioned constraint also holds for it. Since the population size is  $O(n \cdot (\log n + \log W_{max}))$  (Lemma 11), this 1-bit flip happens with a probability of  $\Omega(n^{-2} \cdot (\log n + \log W_{max})^{-1})$  and  $x'$  is obtained in expected time of  $O(n^3 \cdot (\log n + \log W_{max})^2)$ . This new solution will be added to  $P$  because a solution  $y$  with  $Cost(y) + 2 \cdot LP(y) > 2 \cdot OPT$  can not dominate  $x'$  with  $Cost(x') + 2 \cdot LP(x') \leq 2 \cdot OPT$ , and  $x'$  has the minimum value of  $b_2$  among solution that fulfil the constraint. Moreover, if there already is a solution,  $x_{prev}$ , in the same box as  $x'$ , it will be replaced by  $x'$  because  $Cost(x_{prev}) + 2 \cdot LP(x_{prev}) > 2 \cdot OPT$ ; otherwise, it would have been selected as  $x$ .

There are at most  $A = 1 + \lceil \frac{\log n + \log W_{max}}{\log(1+\delta)} \rceil$  different values for  $b_2$  in the objective space, and since  $\delta = \frac{1}{2n}$ ,  $A = O(n \cdot (\log n + \log W_{max}))$ . Therefore, the expected time until a solution  $x''$  is found so that  $b_2(x'') = 0$  and  $Cost(x'') + 2 \cdot LP(x'') \leq 2 \cdot OPT$ , is at most  $O(n^3 \cdot (\log n + \log W_{max})^2)$ . ■

## 5 Diverse Population-based EA

In this section, we introduce a population-based algorithm (see Algorithm 4) that keeps for each  $k$ ,  $0 \leq k \leq n$ , at most two solutions. This implies that the population size is upper bounded by  $2n$ . The two solutions kept in the population are chosen according to different weighing of the cost and the LP-value. For each solution  $x$ , let  $|x|_1$  be the number of selected vertices in  $x$ . Algorithm 4 keeps a new solution  $x'$  in the population, if it minimizes  $Cost(z) + LP(z)$  or  $Cost(z) + 2 \cdot LP(z)$  among other solutions  $x \in P$  where  $|x|_1 = |x'|_1$ . Algorithm 4 gives a detailed description.

Taking into account that the population size is upper bounded by  $2n$  and considering in each step an individual with the smallest number of ones in the population for mutation, one can obtain the following lemma by standard fitness level arguments.

**Lemma 15.** *The search point  $0^n$  is included in the population in expected time of  $O(n^2 \log n)$ .*

To show the main result for Diverse Population-Based EA, we will use the following lemma.

**Lemma 16.** *A solution  $x$  fulfilling the two properties*

1.  $LP(x) = LP(0^n) - Cost(x)$  and
2. *there is an optimal solution of the LP for  $G(x)$  which assigns 1/2 to each non-isolated vertex of  $G(x)$*

*is included in the population of the Diverse Population-Based EA in expected time  $O(n^3)$ .*

*Proof.* By Lemma 15, solution  $0^n$  is contained in the population in expected time  $O(n^2 \log n)$ , which satisfies the property 1 given above. Let  $P' \subseteq P$  be a set containing all solutions in  $P$  that satisfy the property 1 given above.

Let  $x_{max}$  be the solution of  $P'$  with the maximal number of 1-bits. If the optimal fractional vertex cover for  $G(x_{max})$  assigns 1/2 to each non-isolated vertex of  $G(x_{max})$ , then the second property also holds. If the optimal fractional vertex cover for  $G(x_{max})$  assigns 1 to some non-isolated vertex, say  $v$ , then the algorithm selects  $x_{max}$  and flips exactly the bit corresponding to  $v$  with probability  $\Omega(\frac{1}{n^2})$ . Let  $x'$  be the new solution. By selection of  $x_{max}$  we know that  $x'$  is the only solution with  $|x_{max}|_1 + 1$  one-bits; hence, added to  $P$ .

Since the maximum value of  $|x|_1$  is  $n$ , after expected time of  $O(n^3)$ , there is a solution in the population that fulfils the properties given in the lemma. ■

We now show the main result for the Diverse Population-Based EA.

**Theorem 17.** *The expected time until Diverse Population-Based EA has obtained a solution that has approximation ratio  $(1 + \varepsilon)$  is  $O(n \cdot 2^{\min\{n, 2^{(1-\varepsilon)OPT}\}} + n^3)$ .*

*Proof.* By Lemma 16 we know that after expected time of  $O(n^3)$ , there is a solution,  $x$ , in the population that fulfils the properties given in that lemma. With analysis similar to what we had in Theorem 10, we can show that a solution  $x$  with  $Cost(x) + 2 \cdot LP(x) \leq (1 + \varepsilon) \cdot OPT$  is produced in expected time  $O(n \cdot 2^{\min\{n, 2^{(1-\varepsilon)OPT}\}} + n^3)$ .

Now we see whether solution  $x$  is added to population  $P$ . If  $x$  could not be added to  $P$ , then there exists a solution  $y \in P$  such that  $|y|_1 = |x|_1$  and  $Cost(y) + 2 \cdot LP(y) \leq Cost(x) + 2 \cdot LP(x)$ . Thus, the population already includes a solution  $y$  such that  $Cost(y) + 2 \cdot LP(y) \leq (1 + \varepsilon) \cdot OPT$ .

Let  $P'$  be a set containing all solutions  $x \in P$  such that  $Cost(x) + 2 \cdot LP(x) \leq (1 + \varepsilon) \cdot OPT$ . Let  $x_{max} \in P'$  such that  $|x_{max}|_1 = \max_{x \in P'} |x|_1$ .

If  $LP(x_{max}) = 0$ , then solution  $x_{max}$  leads to a vertex cover for graph  $G$ . If  $LP(x_{max}) > 0$ , we present a way to construct a  $(1 + \varepsilon)$ -approximate vertex cover as follows, using  $x_{max}$ . If  $LP(x_{max}) > 0$ , then there exists at least one vertex  $v$  to which the optimal fractional vertex cover  $LP(x_{max})$  assigns value at least 1/2. Then the algorithm selects the solution  $x_{max}$  and flips exactly the bit corresponding to the vertex  $v$  with probability  $\Omega(\frac{1}{n^2})$ . Let  $y$  be the new solution. We have

$$Cost(y) + 2 \cdot LP(y) \leq Cost(x_{max}) + 2 \cdot LP(x_{max}) \leq (1 + \varepsilon) \cdot OPT.$$

Suppose that  $y$  could not be included in  $P$ , then there exists a solution  $y'$  in  $P$  such that  $|y'|_1 = |y|_1$  and  $2 \cdot LP(y') + Cost(y') \leq 2 \cdot LP(y) + Cost(y) \leq (1 + \varepsilon) \cdot OPT$ , which contradicts the assumption that  $|x_{max}|_1 = \max_{x \in P'} |x|_1$ . Therefore, solution  $y$  could be included in  $P$ .

Observe that for any solution  $x$ , if  $|x|_1 = n$ , then  $LP(x) = 0$ . Thus, after expected time of at most  $O(n^3)$ , the population  $P$  could include a solution  $y$  such that  $Cost(y) + 2 \cdot LP(y) \leq (1 + \varepsilon) \cdot OPT$  and  $LP(y) = 0$ , which is a  $(1 + \varepsilon)$ -approximate weighted vertex cover.

Overall, the expected time in which Diverse Population-Based EA finds a  $(1 + \varepsilon)$ -approximate weighted vertex cover, is bounded by  $O(n \cdot 2^{\min\{n, 2(1-\varepsilon)OPT\}} + n^3)$ . ■

## 6 Conclusion

The minimum vertex cover problem is one of the classical NP-hard combinatorial optimization problems. In this paper, we have generalized previous results of Kratsch and Neumann [12] for the unweighted minimum vertex cover problem to the weighted case where in addition weights on the vertices are given. Based on the conference version of this paper [19], in sections 3.2 and 4, we have investigated Global SEMO with alternative mutation operator for finding a  $(1 + \varepsilon)$ -approximation, and studied the algorithm DEMO using the  $\varepsilon$ -dominance approach showing that it reaches a 2-approximation in expected polynomial time. Furthermore, in this paper we have shown that Global SEMO with standard mutation operator efficiently computes a 2-approximation as long as the value of an optimal solution is small. We have also presented a population-based approach with a specific diversity mechanism that reaches an  $(1 + \varepsilon)$ -approximation in expected time  $O(n \cdot 2^{\min\{n, 2(1-\varepsilon)OPT\}} + n^3)$ .

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