Appendix: A Fast Semidefinite Approach to Solving Binary Quadratic Problems

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1. Results on the bound and solution quality of SDCut

We demonstrate some results on the bound of SDCut and the impacts of the parameter σ on the solution quality.

Let us introduce some notations. The objective function and optimal variables of (6) are denoted as $\mathrm{f}(\mathbf{X})$ and $\mathbf{X}^\star$ respectively. $\mathrm{f}_\sigma(\mathbf{X})$ and $\mathbf{X}^\star_\sigma$ represent the objective function and the optimal variables of (11) with the parameter σ, respectively.

Because $\|\mathbf{X}\|_F^2 - \eta^2 \leq 0, \forall\, \mathbf{X} \in \Omega$, the optimal objective value of problem (11) is not larger than that of problem (6) (and its equivalent form (9)), which therefore can be seen as a less tight lower-bound of the optimal objective value of problem (3) (and its equivalent form (5)) compared to the conventional SDP method.

First, we show that the bound given by (11) can be arbitrarily close to the bound given by (6).

Claim A.1. $\forall\, \epsilon > 0, \exists\, \sigma > 0$, *which makes the gap between the optimal objective values of (11) and (6) smaller than* ϵ.

Proof. Based on the definition of (11) and (6), we have:

$$\mathrm{f}(\mathbf{X}) - \mathrm{f}_\sigma(\mathbf{X}) = \sigma(\eta^2 - \|\mathbf{X}\|_F^2) \leq \sigma\eta^2. \quad \text{(A1)}$$

Set σ to ϵ/η^2, then $\mathrm{f}(\mathbf{X}) - \mathrm{f}_\sigma(\mathbf{X}) \leq \epsilon$. Therefore,

$$\mathrm{f}(\mathbf{X}^\star) - \mathrm{f}_\sigma(\mathbf{X}^\star_\sigma) \leq \mathrm{f}(\mathbf{X}^\star_\sigma) - \mathrm{f}_\sigma(\mathbf{X}^\star_\sigma) \leq \epsilon. \quad \text{(A2)}$$

□

Next, we demonstrate the monotonicity of the bound $\mathrm{f}_\sigma(\mathbf{X}^\star_\sigma)$, the conventional SDP objective value $\langle \mathbf{X}^\star_\sigma, \mathbf{A}\rangle$ and the Frobenius norm of $\mathbf{X}^\star_\sigma$, with respect to σ.

Claim A.2. *For* $\sigma_1 \geq \sigma_2 > 0$, *we have (a):* $\mathrm{f}_{\sigma_1}(\mathbf{X}^\star_{\sigma_1}) \leq \mathrm{f}_{\sigma_2}(\mathbf{X}^\star_{\sigma_2})$*; (b):* $\|\mathbf{X}^\star_{\sigma_1}\|_F^2 \leq \|\mathbf{X}^\star_{\sigma_2}\|_F^2$*; (c):* $\langle \mathbf{X}^\star_{\sigma_1}, \mathbf{A}\rangle \geq \langle \mathbf{X}^\star_{\sigma_2}, \mathbf{A}\rangle$.

Proof. (a): First we have

$$\mathrm{f}_{\sigma_1}(\mathbf{X}) - \mathrm{f}_{\sigma_2}(\mathbf{X}) = (\sigma_1 - \sigma_2)(\|\mathbf{X}\|_F^2 - \eta^2) \leq 0. \quad \text{(A3)}$$

Then,

$$\mathrm{f}_{\sigma_1}(\mathbf{X}^\star_{\sigma_1}) - \mathrm{f}_{\sigma_2}(\mathbf{X}^\star_{\sigma_2}) \leq \mathrm{f}_{\sigma_1}(\mathbf{X}^\star_{\sigma_2}) - \mathrm{f}_{\sigma_2}(\mathbf{X}^\star_{\sigma_2}) \leq 0. \quad \text{(A4)}$$

(b): Explicitly, we have

$$\mathrm{f}_{\sigma_1}(\mathbf{X}^\star_{\sigma_1}) \leq \mathrm{f}_{\sigma_1}(\mathbf{X}^\star_{\sigma_2}), \quad \text{(A5)}$$

$$\text{and } \mathrm{f}_{\sigma_2}(\mathbf{X}^\star_{\sigma_1}) \geq \mathrm{f}_{\sigma_2}(\mathbf{X}^\star_{\sigma_2}). \quad \text{(A6)}$$

Then,

$$\begin{aligned} &\mathrm{f}_{\sigma_1}(\mathbf{X}^\star_{\sigma_1}) - \mathrm{f}_{\sigma_2}(\mathbf{X}^\star_{\sigma_1}) \leq \mathrm{f}_{\sigma_1}(\mathbf{X}^\star_{\sigma_2}) - \mathrm{f}_{\sigma_2}(\mathbf{X}^\star_{\sigma_2}) \\ \Rightarrow\ &(\sigma_1 - \sigma_2)(\|\mathbf{X}^\star_{\sigma_1}\|_F^2 - \eta^2) \leq (\sigma_1 - \sigma_2)(\|\mathbf{X}^\star_{\sigma_2}\|_F^2 - \eta^2) \\ \Rightarrow\ &\|\mathbf{X}^\star_{\sigma_1}\|_F^2 \leq \|\mathbf{X}^\star_{\sigma_2}\|_F^2. \quad \text{(A7)} \end{aligned}$$

(c): Still based on (A5) and (A6), we have

$$\begin{aligned} &\mathrm{f}_{\sigma_1}(\mathbf{X}^\star_{\sigma_1}) - \frac{\sigma_1}{\sigma_2}\mathrm{f}_{\sigma_2}(\mathbf{X}^\star_{\sigma_1}) \leq \mathrm{f}_{\sigma_1}(\mathbf{X}^\star_{\sigma_2}) - \frac{\sigma_1}{\sigma_2}\mathrm{f}_{\sigma_2}(\mathbf{X}^\star_{\sigma_2}) \\ \Rightarrow\ &(1 - \frac{\sigma_1}{\sigma_2}) \cdot \langle \mathbf{X}^\star_{\sigma_1}, \mathbf{A}\rangle \leq (1 - \frac{\sigma_1}{\sigma_2}) \cdot \langle \mathbf{X}^\star_{\sigma_2}, \mathbf{A}\rangle \\ \Rightarrow\ &\langle \mathbf{X}^\star_{\sigma_1}, \mathbf{A}\rangle \geq \langle \mathbf{X}^\star_{\sigma_2}, \mathbf{A}\rangle. \quad \text{(A8)} \end{aligned}$$

□

Claim A.1 and Claim A.2 have been varified by the experimental results shown in Table 1 in the paper. SDCut is evaluated with different σs, on the task of random graph bisection. The bound $\mathrm{f}_\sigma(\mathbf{X}^\star_\sigma)$, the objective value $\langle \mathbf{X}^\star_\sigma, \mathbf{A}\rangle$ and the norm $\|\mathbf{X}^\star_\sigma\|_F^2$ are shown in the table. With the decrease of σ, the lower-bound becomes tigher; Meanwhile, $\langle \mathbf{X}^\star_\sigma, \mathbf{A}\rangle$ decreases and $\|\mathbf{X}^\star_\sigma\|_F^2$ increases monotonically. When $\sigma = 10^{-4}$, the lower-bound (-21.31) is very close to the one given by standard SDP (-21.29).

2. Results on image co-segmentation

In Fig. A1, we demonstrate more results on image co-segmentation. In our experiments, the confidence maps of SDCut and LowRank are similar, while SDCut is faster than LowRank.

Figure A1: Co-segmentation results on Weizman horses and MSRC databases. The original images, the results of LowRank and SDCut are illustrated from top to bottom. LowRank and SDCut have similar confidence maps. σ is set to 10^{-4}.