Supplementary Material for
Paper 1298: Guaranteed Outlier Removal for Rotation Search

A Image Stitching

We present image stitching results (i.e., using optimized rotations to construct homography warps) for 4 additional image pairs; see Figs. A1–A4. The experimental settings are as described in Sec. 5.2 in the main paper. For brevity, we do not include RANSAC results here, since it is clear from the main paper that running RANSAC does not assist in cutting down the overall runtime of globally optimal rotation search.

A.1 Quantitative results

The quantitative results are listed in Table A; consult Sec. 5.2 in the main paper for the definition of the performance metrics.

Image pair	N	irat	GORE				BnB		GORE +BnB
			lwbnd	err (°)	out	time (s)	opt	time (s)	time (s)
machu-picchu	194	0.402	66	0.14	59	0.04	78	1.04	0.68
paris1	718	0.089	61	0.04	641	0.38	64	20.73	1.81
paris2	921	0.275	247	0.06	294	2.55	253	11.49	10.04
rio	675	0.213	139	0.10	366	0.66	144	7.31	3.53

Table A: Image stitching results.

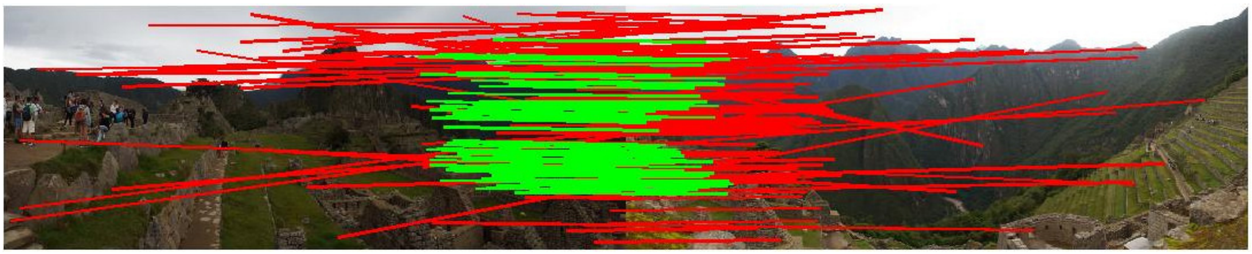
A.2 Qualitative results

Panel (b) in Figs. A1–A4 show the SIFT keypoint matches. Green lines designate the inliers that agree with the globally optimal rotation, while red lines designate "true" outliers with respect to the globally optimal rotation.

Panel (c) show the matches that remain after GORE; observe that the matches that have been removed do not exist in the true inlier set.

To demonstrate that the suboptimal rotation produced by GORE (see Algorithm 1 and Sec. 5.1 in the main paper) is very close to the optimal one, panel (d) show the stitched images using the homographies defined using the suboptimal rotations. Compare with the stitching results in panel (e) where the homographies were defined using the globally optimal rotation found by BnB.

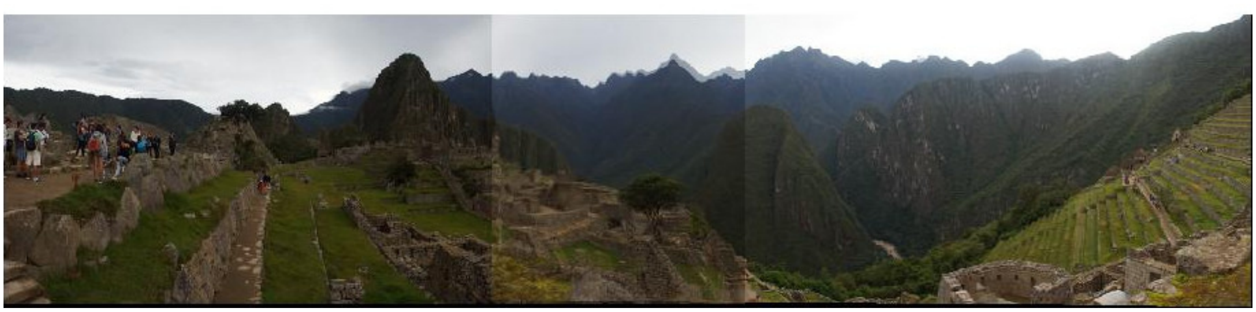
(a) Input images.



(b) SIFT keypoint matches (green = true inliers, red = true outliers).

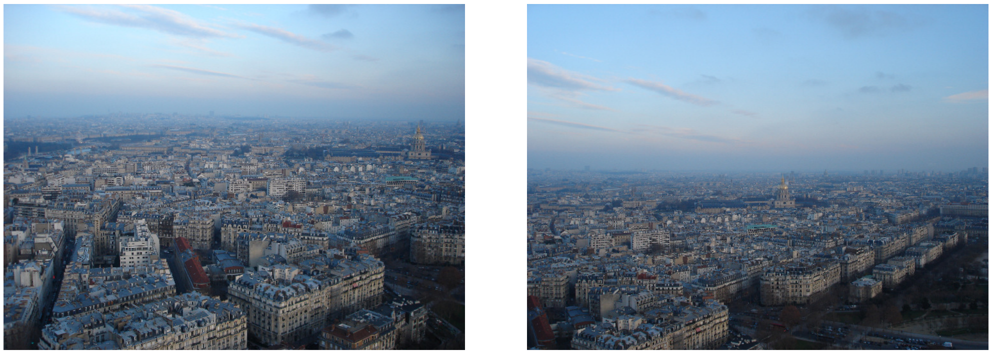
(c) Matches that remain after preprocessing with GORE.

(d) Stitching result using suboptimal rotation by GORE.

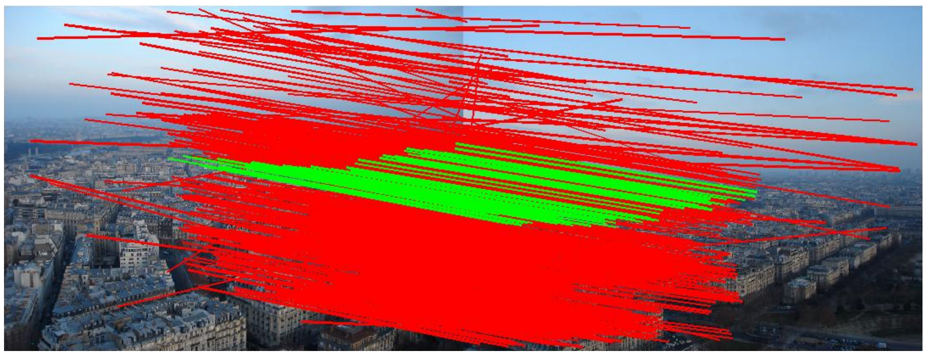


(e) Stitching result using globally optimal rotation by BnB.

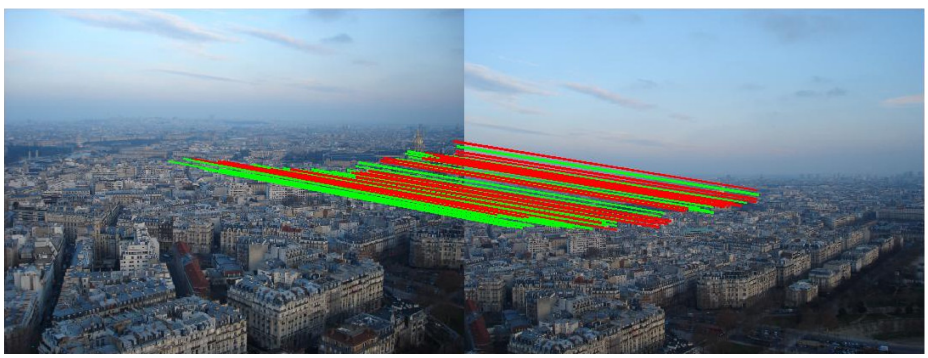
Figure A1: Results for the *machu-picchu* image pair.



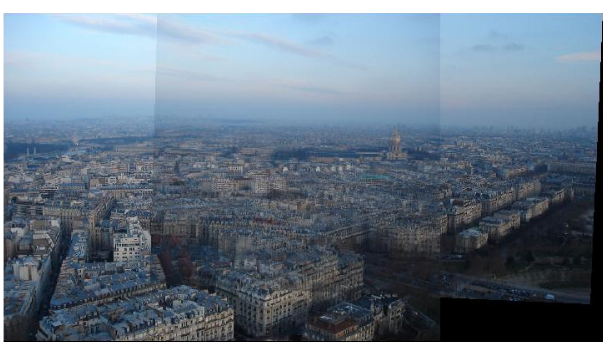
(a) Input images.



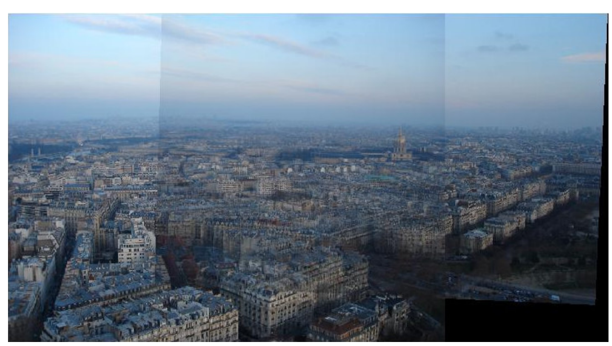
(b) SIFT keypoint matches (green = true inliers, red = true outliers).



(c) Matches that remain after preprocessing with GORE.



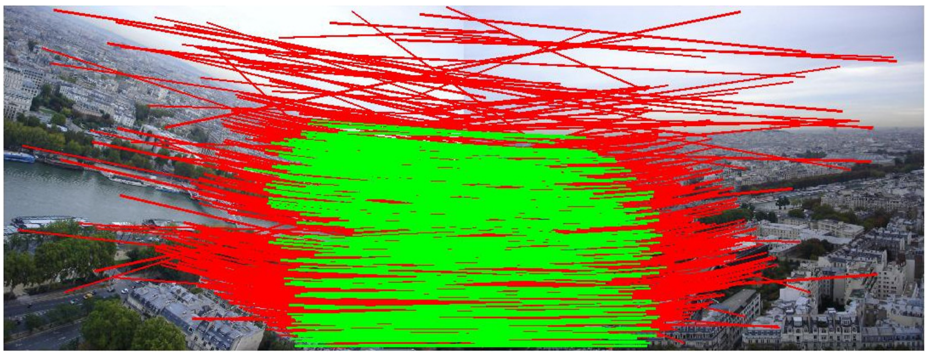
(d) Stitching result using suboptimal rotation by GORE.



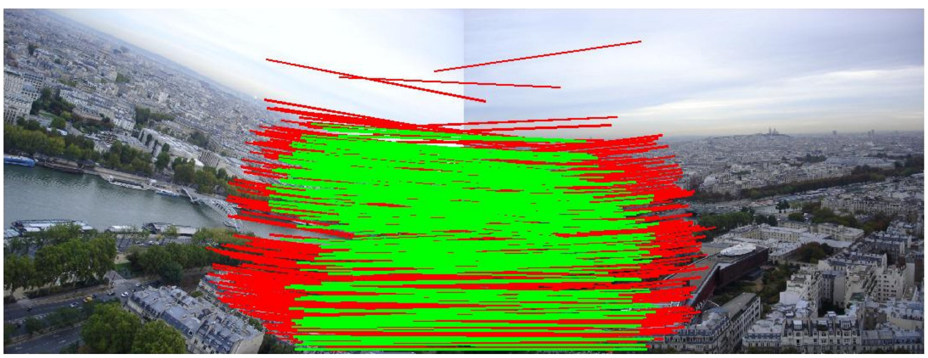
(e) Stitching result using globally optimal rotation by BnB.

Figure A2: Results for the *paris1* image pair.

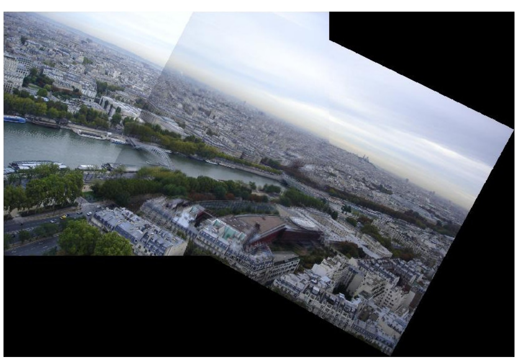
(a) Input images



(b) SIFT keypoint matches (green = true inliers, red = true outliers).



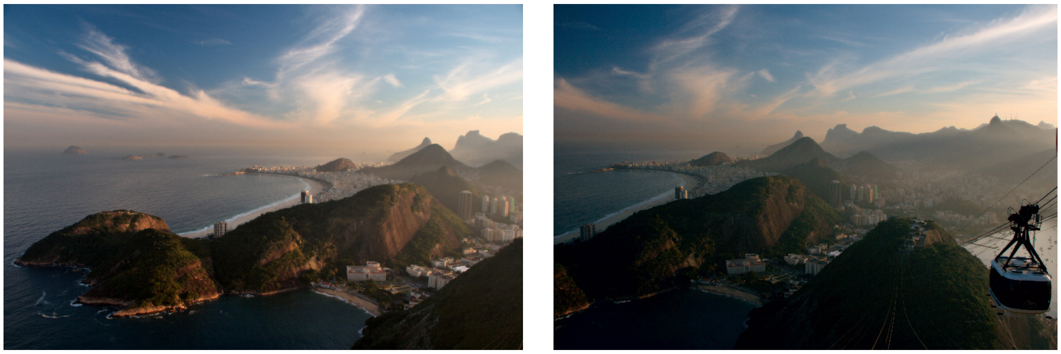
(c) Matches that remain after preprocessing with GORE.



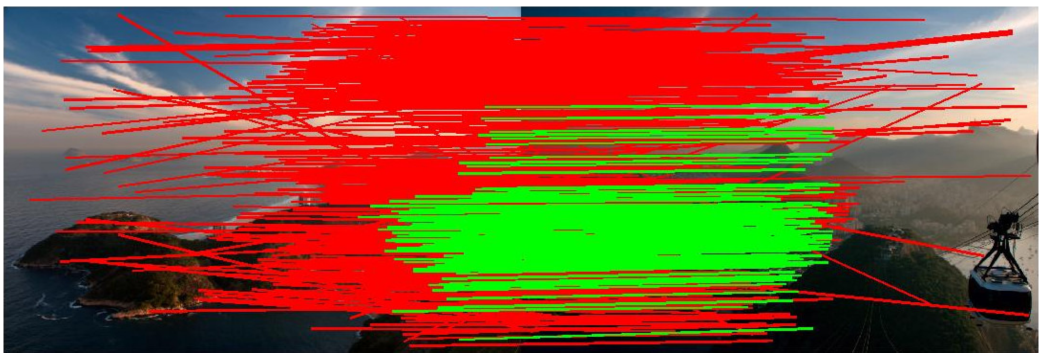
(d) Stitching result using suboptimal rotation by GORE.

(e) Stitching result using globally optimal rotation by BnB.

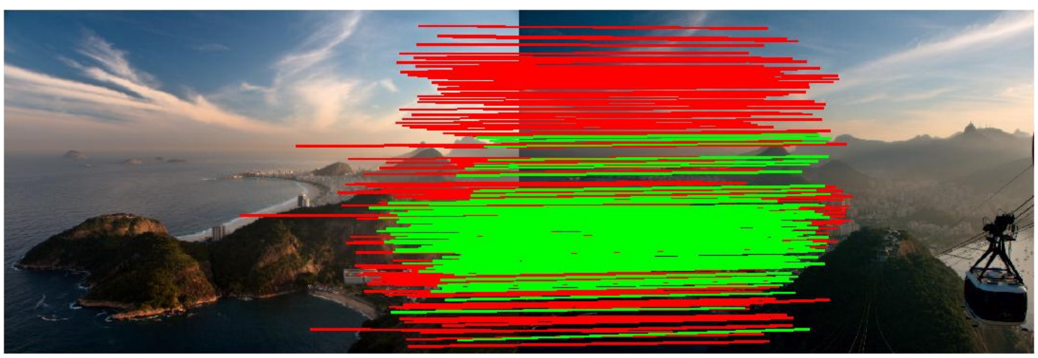
Figure A3: Results for the *paris2* image pair.



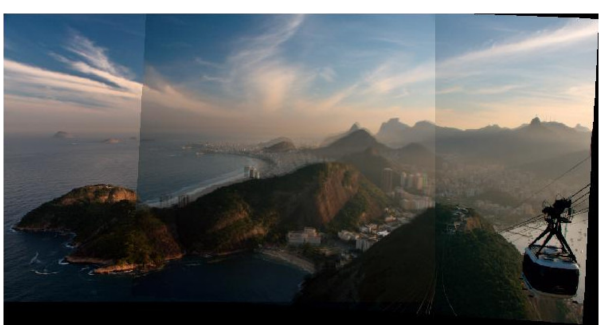
(a) Input images



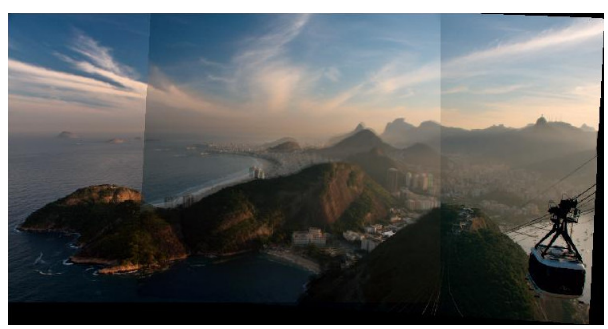
(b) SIFT keypoint matches (green = true inliers, red = true outliers).



(c) Matches that remain after preprocessing with GORE.



(d) Stitching result using suboptimal rotation by GORE.



(e) Stitching result using globally optimal rotation by BnB.

Figure A4: Results for the *rio* image pair.

B Bounding α_i and β_i

In this section, we detail the method used to find upper bounds for solutions of equations in (31) in the main paper. For simplicity of notation, we wrote equations from (31) in main paper without indices as

$$\sin(\alpha) = \frac{\sin(\delta(\theta^a))}{c}, \quad \sin(\beta) = \frac{\sin(\delta(\theta^b))}{c} \tag{1}$$

where $c = \sin(\psi(\mathbf{x}))$. Following the same change of notation, we re-write equations (29) and (30)

$$\delta(\theta^a) = 2|\rho^a - \alpha| \sin(\epsilon/2) + \epsilon \quad \text{and} \tag{2}$$

$$\delta(\theta^b) = 2|\rho^b + \beta| \sin(\epsilon/2) + \epsilon \tag{3}$$

where $\rho^a = \theta - \gamma$ and $\rho^b = \theta + \gamma$.

The presented method (depicted in Fig. A5) to find upper bounds α' and β' of the solutions α^* and β^* of equations in (1) consist in solving the intersection of the two-piece-linear function $g\colon [0, \pi/2) \to [0, 1]$

$$g(t) = \begin{cases} \frac{2\sqrt{2}}{\pi} t & \text{if } 0 \leq t < \pi/4 \\ \frac{2(2-\sqrt{2})}{\pi} t + \sqrt{2} - 1 & \text{if } \pi/4 \leq t < \pi/2 \end{cases} \tag{4}$$

with $\delta(\theta^a)/c$ and $\delta(\theta^b)/c$ for α' and β' resp., when $\rho^a \geq 0$ and $\rho^b \leq \pi$. If $\rho^a < 0$ or $\rho^b > \pi$, α' and β' are obtained by solving a mirrored problem. In the rest of this supplementary material we will show that the introduced method guarantees that $\alpha' \geq \alpha^*$ and $\beta' \geq \beta^*$ and we will give details of how fix bounds for border cases.

B.1 Range of α_i and β_i

Definitions $\delta(\theta^a)$ and $\delta(\theta^b)$ in equations (2) and (3) are consequence of the well-known bound over the axis-angle representation, thus $\delta(\theta^a)$ (resp. $\delta(\theta^b)$) is maximized when $|\rho^a - \alpha|$ (resp. $|\rho^b + \beta|$) is equal to π.

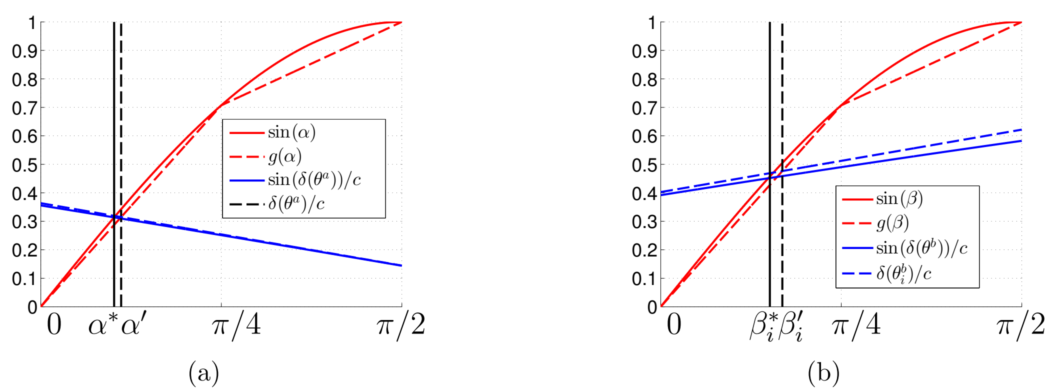


Figure A5: Solving α and β using linear approximations.

We solve first α' and β' when $\rho^a \geq 0$ and $\rho^b \leq \pi$. We can rewrite the original problem (1) as

$$\alpha = \arcsin\left(\frac{\sin(\delta(\theta^a))}{c}\right), \quad \beta = \arcsin\left(\frac{\sin(\delta(\theta^b))}{c}\right). \tag{5}$$

Then, α and β have solution if $c > 0$ and

$$\sin(\delta(\theta^a)) \leq c \qquad \text{and} \qquad \sin(\delta(\theta^b)) \leq c \tag{6}$$

as arcsin yields non negative values when evaluated in the range $[0, 1]$. Note that $c \in [0, 1]$ as $c = \sin(\psi(\mathbf{x}))$ for $\psi(\mathbf{x}) \in [0, \pi]$. Note also that $\delta(\theta^a)$ and $\delta(\theta^b)$ are $\geq \epsilon$. This result shows that equations in (5) will have solution depending on the inclination $\psi(\mathbf{x})$. To find a bound over $\psi(\mathbf{x})$ that guarantees the existence of solution for equations in 5, we apply the Jordan's inequality

$$\frac{2}{\pi}t \leq \sin(t) \leq t \ \text{ for } \ t \leq \frac{\pi}{2} \tag{7}$$

to bound results in 6, then

$$2\pi \sin(\epsilon/2) + \epsilon \leq c \tag{8}$$

if $\epsilon < \frac{\pi}{2(\pi+1)} \equiv 21.7°$.

By using again Jordan's inequality and replacing c by its definition, we establish the bound over the inclination

$$\epsilon(\pi + 1) \leq \sin(\psi(\mathbf{x})). \tag{9}$$

As the inclination $\psi(\mathbf{x})$ is defined in $[0, \pi]$, we cannot apply Jordan's inequality directly. However, instead of measuring inclination we can measure the angular distance to the "closest Pole" $\lambda(\mathbf{x})$ which is defined in $[0, \pi/2]$ and with $\sin(\lambda(\mathbf{x})) = \sin(\psi(\mathbf{x}))$, then applying Jordan's inequality we can bound $\lambda(\mathbf{x})$ by

$$\lambda(\mathbf{x}) \geq \epsilon\frac{\pi(\pi + 1)}{2} \equiv 6.506\epsilon. \tag{10}$$

Previous result shows that there exist solution of equations in 5 if $\mathbf{x}$ is not too close to the poles.

Note that arcsin takes values in $[0, \pi/2]$ when evaluated in the range $[0, 1]$. That range contains the range $[0, \pi/2)$ where α and β are well defined. We will show that β is defined in the range $[0, \pi/2)$ and the same analysis can be performed for α. Let the outline of the spherical region $S_{\delta(\theta^b)}(\mathbf{A}_{\theta^b,\mathbf{y}_k}\hat{\mathbf{B}}\mathbf{x})$ be infinitesimally close to either the North or the South Pole such that it is not included in $S_{\delta(\theta^b)}(\mathbf{A}_{\theta^b,\mathbf{y}_k}\hat{\mathbf{B}}\mathbf{x})$. In such a limit configuration, the meridians containing $S_{\delta(\theta^b)}(\mathbf{A}_{\theta^b,\mathbf{y}_k}\hat{\mathbf{B}}\mathbf{x})$ will be at azimuthal angle π apart, then β is bounded by $\pi/2$. If either the North or the South Pole is contained in $S_{\delta(\theta^b)}(\mathbf{A}_{\theta^b,\mathbf{y}_k}\hat{\mathbf{B}}\mathbf{x})$, then that spherical region cannot be bounded between two particular meridians as the spherical region will intersect points over the sphere for all the azimuthal range $[-\pi, \pi]$.

B.2 Validity of the proposed method

Note that Result (10) ensures that $S_{\delta(\theta^a)}(\mathbf{A}_{\theta^a,\mathbf{y}_k}\hat{\mathbf{B}}\mathbf{x})$ and $S_{\delta(\theta^b)}(\mathbf{A}_{\theta^b,\mathbf{y}_k}\hat{\mathbf{B}}\mathbf{x})$ are enough far from the North and the South Pole to establish the interval $[\theta^a, \theta^b]$.

We aim to find upper bounds α' and β' as there is no close form to solve equations in (5). Note that the intersection of the identity and the corresponding RHS solves equations in (5). We name RHSs in equations in (5) as f^a and f^b. Note that $y_0^a := f^a(0)$ and $y_0^b := f^b(0)$ are equal to

$$y_0^a = \arcsin\left(\frac{\sin\left(2\rho^a \sin(\epsilon/2) + \epsilon\right)}{c}\right) \qquad y_0^b = \arcsin\left(\frac{\sin\left(2\rho^b \sin(\epsilon/2) + \epsilon\right)}{c}\right). \tag{11}$$

We aim to show that y_0^a and y_0^b are non negative. That is clearly true as arcsin is non negative in the range $[0, 1]$, and c, $\sin(2\rho^a \sin(\epsilon/2) + \epsilon)$ and $\sin(2\rho^b \sin(\epsilon/2) + \epsilon)$ are > 0. If inequality (10) is true, then $\sin(\delta(\theta^a))/c$ and $\sin(\delta(\theta^b))/c$ are < 1 as this is the condition for f^a and f^b to have solution in $[0, \pi/2)$. Let $\hat{f}^a$ and $\hat{f}^b$ be upper bounds of f^a and f^b, and $\hat{\alpha}$ and $\hat{\beta}$ such as $\hat{\alpha} = f^a(\hat{\alpha})$ and $\hat{\beta} = f^b(\hat{\beta})$.

If $\hat{\alpha}$ or $\hat{\beta}$ are $\geq \pi/2$ we can abort trying to find α' and β', and establish the interval $[\theta^a, \theta^b] = [-\pi, \pi]$. Otherwise α^* and β^* must be in $[0, \hat{\alpha}]$ and $[0, \hat{\beta}]$. To be clear, f^a must intersects the identity for some α in the range $[0, \hat{\alpha}]$ as f^a is a continuous function and there exist two points in the range $[0, \hat{\alpha}]$ at opposites sides of the identity. Specifically, for points 0 and $\hat{\alpha}$

$$f^a(0) > 0 \qquad \text{and} \qquad f^a(\hat{\alpha}) \leq \hat{f}^a(\hat{\alpha}) = \hat{\alpha}. \tag{12}$$

Then, by solving $\hat{\alpha}$ we are able to bound α^*. The same analysis can be performed for $\hat{\beta}$.

Upper bounds f^a and f^b can be simple taken as

$$\hat{f}^a(\alpha) = \arcsin\left(\frac{2|\rho^a - \alpha| \sin(\epsilon/2) + \epsilon}{c}\right) \tag{13}$$

$$\hat{f}^b(\beta) = \arcsin\left(\frac{2|\rho^b + \beta| \sin(\epsilon/2) + \epsilon}{c}\right) \tag{14}$$

Note that bounds are valid if

$$2|\rho^a - \alpha| \sin(\epsilon/2) + \epsilon \leq \pi/2 \tag{15}$$

$$2|\rho^b + \beta| \sin(\epsilon/2) + \epsilon \leq \pi/2 \tag{16}$$

according to the Jordan's inequality. That is equivalent to $\epsilon \leq \pi/(2\pi + 2) \equiv 21.7°$.

To finally find α' and β' we formulate new equations which solutions are upper bounds of α^* and β^*

$$\sin(\alpha) = \frac{2|\rho^a - \alpha| \sin(\epsilon/2) + \epsilon}{c} \qquad \text{and} \qquad \sin(\beta) = \frac{2|\rho^b + \beta| \sin(\epsilon/2) + \epsilon}{c}. \tag{17}$$

The solutions for the defined equations in (17) correspond to the intersection of a rect with the sin function if $\alpha \leq \rho^a$, which is equivalent to $\theta^a \geq 0$. Similarly, if $\beta \geq \pi - \rho^b$, is equivalent to $\theta^b \geq \pi$.

To simplify equations in (17) we search for α and β that conduct the interval $[\theta^a, \theta^b]$ be contained in $[0, \pi]$. Then we aim to solve

$$\sin(\alpha) = m^a\alpha + b^a \qquad \text{and} \qquad \sin(\beta) = m^b\beta + b^b \tag{18}$$

where

$$m^a = -\frac{2\sin(\epsilon/2)}{c} \qquad m^b = \frac{2\sin(\epsilon/2)}{c} \qquad b^a = \frac{2\rho^a\sin(\epsilon/2)+\epsilon}{c} \qquad b^b = \frac{2\rho^b\sin(\epsilon/2)+\epsilon}{c} \tag{19}$$

If by solving equations in (18), α or β induces $\theta^a < 0$ or $\theta^b > \pi$, α and β can be fixed with

$$\alpha_{\text{min}} = \arcsin\left(\frac{\sin(2\theta\sin(\epsilon/2)+\epsilon)}{c}\right) \qquad \text{and} \tag{20}$$

$$\beta_{\text{max}} = \min\left(\arcsin\left(\frac{\sin(2\pi\sin(\epsilon/2)+\epsilon)}{c}\right), \beta'\right) \tag{21}$$

without producing invalid bounds.

Since there is no closed form to equations (18), we obtain the bounds α' and β' as upper bounds of its solution. We proceed in a similar fashion that when defining equations (17), i.e., we solve new formulations which solutions are greater than original equations. As sin is concave in $[0, \pi/2)$, function g (defined in (4)) is a lineal lower bound of sin in $[0, \pi/2)$. If α' and β' solve the intersection of g with RHSs of equations in (18), and α' and β' are $< \pi/2$, then the solution of equations in (18) must be in $[0, \alpha']$ and $[0, \beta']$.

It remains to solve the case when $\rho^a < 0$ (resp. $\rho^b > \pi$). However, it is a mirrored problem of when $\rho^a \geq 0$ (resp. $\rho^b \leq \pi$) and α' (resp. β') can be obtained as it were β (resp. α) by solving for $\rho^a_{\text{mirror}} = -\rho^a$ (resp. $\rho^b_{\text{mirror}} = 2\pi - \rho^b$).