Vector Algebra and Calculus

1. Revision of vector algebra, scalar product, vector product
2. Triple products, multiple products, applications to geometry
3. Differentiation of vector functions, applications to mechanics
4. Scalar and vector fields. Line, surface and volume integrals, curvilinear co-ordinates
5. Vector operators — grad, div and curl
6. **Vector Identities, curvilinear co-ordinate systems**
7. Gauss' and Stokes' Theorems and extensions
8. Engineering Applications

6. Vector Operator Identities & Curvi Coords

- In this lecture we look at identities built from vector operators.
- These operators behave both as vectors and as differential operators, so that the usual rules of taking the derivative of, say, a product must be observed.
- We are laying the groundwork for the use of these identities in later parts of the Engineering course.

- We then turn to derive expressions for grad, div and curl in curvilinear coordinates.
- After deriving general expressions, we will specialize to the Polar family.

Identity 1: curl grad $U = \mathbf{0}$

- $U(x, y, z)$ is a scalar field.

  Then

$$
\begin{aligned}
\nabla \times \nabla U &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial U/\partial x & \partial U/\partial y & \partial U/\partial z \end{vmatrix} \\
&= \hat{\mathbf{i}} \left(\frac{\partial^2 U}{\partial y \partial z} - \frac{\partial^2 U}{\partial z \partial y} \right) + \hat{\mathbf{j}}\,() + \hat{\mathbf{k}}\,() \\
&= \mathbf{0} \quad .
\end{aligned}
$$

- $\nabla \times \nabla$ can be thought of as a null operator.

Identity 2: div curl $\mathbf{a} = 0$

- For $\mathbf{a}(x, y, z)$ a vector field:

$$
\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{a}) &= \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix} \\
&= \quad \frac{\partial^2 a_z}{\partial x \partial y} - \frac{\partial^2 a_y}{\partial x \partial z} \\
&\qquad - \frac{\partial^2 a_z}{\partial y \partial x} + \frac{\partial^2 a_x}{\partial y \partial z} \\
&\qquad\quad + \frac{\partial^2 a_y}{\partial z \partial x} - \frac{\partial^2 a_x}{\partial z \partial y} \\
&= 0
\end{aligned}
$$

Identity 3: divergence of $U\mathbf{v}$

- Suppose that
  - $U(\mathbf{r})$ is a scalar field
  - $\mathbf{v}(\mathbf{r})$ is a vector field

  and we are interested in the divergence of the product $U\mathbf{v}$.

- For example
  - $U(\mathbf{r})$ could be fluid density; and
  - $\mathbf{v}(\mathbf{r})$ its instantaneous velocity

  The product would be the mass flux per unit area.

- The product $U\mathbf{v}$ is a vector field, so we can compute its divergence ...

$$\nabla \cdot (U\mathbf{v}) = U(\nabla \cdot \mathbf{v}) + (\nabla U) \cdot \mathbf{v} = U\text{div}\mathbf{v} + (\text{grad}U) \cdot \mathbf{v}$$

- In steps:

$$\begin{aligned} \nabla \cdot (U\mathbf{v}) &= \left(\frac{\partial}{\partial x}(Uv_x) + \frac{\partial}{\partial y}(Uv_y) + \frac{\partial}{\partial z}(Uv_z) \right) \\ &= U\frac{\partial v_x}{\partial x} + U\frac{\partial v_y}{\partial y} + U\frac{\partial v_z}{\partial z} + v_x\frac{\partial U}{\partial x} + v_y\frac{\partial U}{\partial y} + v_z\frac{\partial U}{\partial z} \\ &= U\text{div}\mathbf{v} + \mathbf{v} \cdot \text{grad}U \end{aligned}$$

Identity 3: curl of $U\mathbf{a}$

- In a similar way, we can take the curl of the product of a scalar and vector field field $U\mathbf{v}$.
- The result should be a vector field.
- And you're probably happy now to write down

$$\nabla \times (U\mathbf{v}) = U(\nabla \times \mathbf{v}) + (\nabla U) \times \mathbf{v} \quad .$$

Identity 4: div of $\mathbf{a} \times \mathbf{b}$

- But things get trickier to guess when vector or scalar products are involved!
- Eg, not at all obvious that:

$$\mathrm{div}(\mathbf{a} \times \mathbf{b}) = \mathrm{curl}\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathrm{curl}\mathbf{b}$$

- To show this, use the determinant:

$$\begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \frac{\partial}{\partial x}[a_y b_z - a_z b_y] + \frac{\partial}{\partial y}[a_z b_x - a_x b_z] + \frac{\partial}{\partial z}[a_x b_y - a_y b_x]$$

$$= \ldots$$

Vector operator identities in HLT

- We could carry on inventing vector identities for some time, but it is a bit, er, dull.
- Why bother at all, as they are in HLT?

  1. Since grad, div and curl describe key aspects of vectors fields, they often arise often in practice.

     The identities can save you a lot of time and hacking of partial derivatives, as we will see when we consider Maxwell's equation as an example later.
  2. Secondly, they help to identify other practically important vector operators.

- We now look at such an example.

Identity 5: curl($\mathbf{a} \times \mathbf{b}$)

$$\text{curl}(\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_y b_z - a_z b_y & a_z b_x - a_x b_z & a_x b_y - a_y b_x \end{vmatrix}$$

$$\Rightarrow \ \text{curl}(\mathbf{a} \times \mathbf{b})_x = \frac{\partial}{\partial y}(a_x b_y - a_y b_x) - \frac{\partial}{\partial z}(a_z b_x - a_x b_z)$$

This can be written as the sum of four terms:

$$a_x \left(\bullet + \frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z} \right) - b_x \left(* + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \right) +$$

$$\left[* + b_y \frac{\partial}{\partial y} + b_z \frac{\partial}{\partial z} \right] a_x - \left[\bullet + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z} \right] b_x$$

$\bullet$: $a_x \frac{\partial b_x}{\partial x}$ add to term1, sub from term4

$*$: $b_x \frac{\partial a_x}{\partial x}$: sub from term2, add to term3

Hence

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\nabla \cdot \mathbf{b})\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{b} + [\mathbf{b} \cdot \nabla]\mathbf{a} - [\mathbf{a} \cdot \nabla]\mathbf{b}$$

$[\mathbf{a} \cdot \nabla]$ can be regarded as new, and very useful, scalar differential operator.

Definition of the operator $[\mathbf{a} \cdot \nabla]$

- This is a *scalar operator* ...

$$[\mathbf{a} \cdot \nabla] \equiv \left[a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z} \right] \quad .$$

- Notice that the components of **a** don't get touched by the differentiation.
- Applied to a scalar field, results in a scalar field
- Applied to a vector field results in a vector field

Identity 6: curl(curl**a**) for you to derive

- Amuse yourself by deriving the following important identity ...

$$\mathrm{curl}(\mathrm{curl}\,\mathbf{a}) = \mathrm{grad}(\mathrm{div}\,\mathbf{a}) - \nabla^2\mathbf{a}$$

where

$$\nabla^2\mathbf{a} = \nabla^2 a_x\hat{\mathbf{\imath}} + \nabla^2 a_y\hat{\mathbf{\jmath}} + \nabla^2 a_z\hat{\mathbf{k}}$$

- We are about to use it ....

♣ Eg using Identity 6: electromagnetic waves

- Background: Maxwell established a set of four vector equations which are fundamental to working out how eletromagnetic waves propagate. The entire telecommunications industry is built on these!

$$\begin{aligned}
\text{div}\mathbf{D} &= \rho \\
\text{div}\mathbf{B} &= 0 \\
\text{curl}\mathbf{E} &= -\frac{\partial}{\partial t}\mathbf{B} \\
\text{curl}\mathbf{H} &= \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D}
\end{aligned}$$

- In addition, we can assume the following

$$\begin{aligned}
\mathbf{B} &= \mu_r\mu_0\mathbf{H} \\
\mathbf{J} &= \sigma\mathbf{E} \\
\mathbf{D} &= \epsilon_r\epsilon_0\mathbf{E}
\end{aligned}$$

Example ctd

Question: Show that in a material with no free charge, $\rho = 0$, and with zero conductivity, $\sigma = 0$, the electric field $\mathbf{E}$ must be a solution of the wave equation $\nabla^2\mathbf{E} = \mu_r\mu_0\epsilon_r\epsilon_0(\partial^2\mathbf{E}/\partial t^2)$.

Answer:

$$
\begin{aligned}
\text{div}\mathbf{D} &= \text{div}(\epsilon_r\epsilon_0\mathbf{E}) = \epsilon_r\epsilon_0\text{div}\mathbf{E} = \rho = 0; \Rightarrow \text{div}\mathbf{E} = 0 \\
\text{div}\mathbf{B} &= \text{div}(\mu_r\mu_0\mathbf{H}) = \mu_r\mu_0\text{div}\mathbf{H} = 0 \quad \Rightarrow \text{div}\mathbf{H} = 0 \\
\text{curl}\mathbf{E} &= -\partial\mathbf{B}/\partial t = -\mu_r\mu_0(\partial\mathbf{H}/\partial t) \\
\text{curl}\mathbf{H} &= \mathbf{J} + \partial\mathbf{D}/\partial t = \mathbf{0} + \epsilon_r\epsilon_0(\partial\mathbf{E}/\partial t)
\end{aligned}
$$

But $\text{curlcurl}\mathbf{E} = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E}$, so

$$
\begin{aligned}
\text{curl}\left[-\mu_r\mu_0(\partial\mathbf{H}/\partial t)\right] &= -\nabla^2\mathbf{E} \\
-\mu_r\mu_0\frac{\partial}{\partial t}[\text{curl}\mathbf{H}] &= -\nabla^2\mathbf{E}
\end{aligned}
$$

Then

$$
\begin{aligned}
-\mu_r\mu_0\frac{\partial}{\partial t}\left[\epsilon_r\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right] &= -\nabla^2\mathbf{E} \\
\Rightarrow \mu_r\mu_0\epsilon_r\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} &= \nabla^2\mathbf{E}
\end{aligned}
$$

- It is possible to obtain general expressions for grad, div and curl in any orthogonal curvilinear co-ordinate system ...
- Need the scale factors h ...
- We recall that the unit vector in the direction of increasing u, with v and w being kept constant, is

$$\hat{\mathbf{u}} = \frac{1}{h_u}\frac{\partial \mathbf{r}}{\partial u}$$

where $\mathbf{r}$ is the general position vector, and

$$h_u = \left|\frac{\partial \mathbf{r}}{\partial u}\right|$$

and similar expressions apply for the other co-ordinate directions. Then

$$d\mathbf{r} = h_u du \hat{\mathbf{u}} + h_v dv \hat{\mathbf{v}} + h_w dw \hat{\mathbf{w}} \ .$$

Grad in curvilinear coordinates

- Using the properties of the gradient of a scalar field obtained previously,

$$\nabla U \cdot d\mathbf{r} = dU \quad and \quad dU = \frac{\partial U}{\partial u}du + \frac{\partial U}{\partial v}dv + \frac{\partial U}{\partial w}dw$$

It follows that

$$\nabla U \cdot (h_u \hat{\mathbf{u}} du + h_v \hat{\mathbf{v}} dv + h_w \hat{\mathbf{w}} dw) = \frac{\partial U}{\partial u}du + \frac{\partial U}{\partial v}dv + \frac{\partial U}{\partial w}dw$$

- The only way this can be satisfied for independent du, dv, dw is when

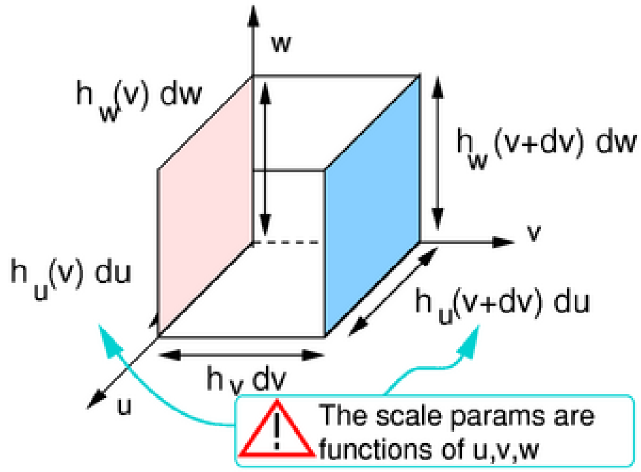
Grad U in curvilinear coords:

$$\nabla U = \frac{1}{h_u}\frac{\partial U}{\partial u}\hat{\mathbf{u}} + \frac{1}{h_v}\frac{\partial U}{\partial v}\hat{\mathbf{v}} + \frac{1}{h_w}\frac{\partial U}{\partial w}\hat{\mathbf{w}}$$

- If the curvilinear coordinates are orthogonal then δvolume is a cuboid (to 1st order in small things) and

$$dV = h_u \; h_v \; h_w \;\; du \; dv \; dw \quad .$$

- However, it is not quite a cuboid: the area of two opposite faces will differ as the scale parameters are functions of u, v, w.



- So the nett efflux from the two faces in the $\hat{\mathbf{v}}$ dirn is

$$= \left[a_v + \frac{\partial a_v}{\partial v} dv\right] \left[h_u + \frac{\partial h_u}{\partial v} dv\right] \left[h_w + \frac{\partial h_w}{\partial v} dv\right] du dw - a_v h_u h_w du dw$$

$$\approx \frac{\partial (a_v h_u h_w)}{\partial v} du dv dw$$

- Repeat: the nett efflux from the two faces in the $\hat{\mathbf{v}}$ dirn is

$$= \left[a_v + \frac{\partial a_v}{\partial v}dv\right]\left[h_u + \frac{\partial h_u}{\partial v}dv\right]\left[h_w + \frac{\partial h_w}{\partial v}dv\right] du dw - a_v h_u h_w du dw$$

$$= \frac{\partial(a_v h_u h_w)}{\partial v} du dv dw$$

- Now div is net efflux per unit volume, so sum up other faces:

$$\operatorname{div}\mathbf{a}\, dV = \left(\frac{\partial(a_u\ h_v\ h_w)}{\partial u} + \frac{\partial(a_v\ h_u\ h_w)}{\partial v} + \frac{\partial(a_w\ h_u\ h_v)}{\partial w}\right)\ du dv dw$$

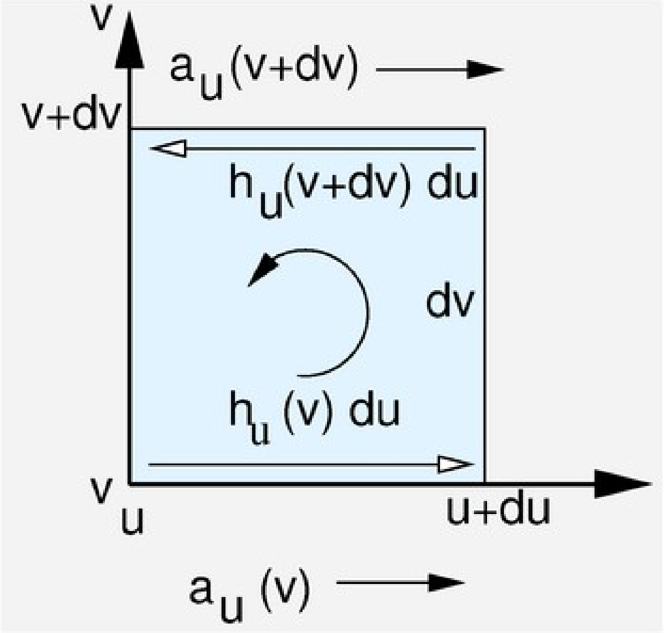
- Then divide by $dV = h_u h_v h_w du dv dw$ ...

Conclude: div in curvi coords is:

$$\operatorname{div}\mathbf{a} = \frac{1}{h_u h_v h_w}\left(\frac{\partial(a_u\ h_v\ h_w)}{\partial u} + \frac{\partial(a_v\ h_u\ h_w)}{\partial v} + \frac{\partial(a_w\ h_u\ h_v)}{\partial w}\right)$$

Curl in curvilinear coordinates

- For an orthogonal curvi coord system $dS = h_u h_v du dw$.
- But the opposite sides are not of same length! Lengths are
  $h_u(v)du$, and $h_u(v + dv)du$.



- Summing this pair contributes to circulation (in $\hat{\mathbf{w}}$ dirn)

$$a_u(v)h_u(v)du - a_u(v + dv)h_u(v + dv)du = -\frac{\partial(h_u a_u)}{\partial v} dv du$$

- Add in the other pair to find circulation per unit area

$$\frac{dC}{h_u h_v du dv} = \frac{1}{h_u h_v}\left(-\frac{\partial(h_u a_u)}{\partial v} + \frac{\partial(h_v a_v)}{\partial u}\right)$$

Curl in curvilinear coordinates, ctd

- To repeat, the part related to $\hat{\mathbf{w}}$ is:

$$\frac{dC}{h_u h_v du dv} = \frac{1}{h_u h_v}\left(-\frac{\partial(h_u a_u)}{\partial v} + \frac{\partial(h_v a_v)}{\partial u}\right)$$

- Adding in the other two components gives:

$$\begin{aligned}\mathrm{curl}\mathbf{a}(u,v,w) \;=\; & \frac{1}{h_v h_w}\left(\frac{\partial(h_w a_w)}{\partial v} - \frac{\partial(h_v a_v)}{\partial w}\right)\hat{\mathbf{u}} \; + \\ & \frac{1}{h_w h_u}\left(\frac{\partial(h_u a_u)}{\partial w} - \frac{\partial(h_w a_w)}{\partial u}\right)\hat{\mathbf{v}} \; + \\ & \frac{1}{h_u h_v}\left(\frac{\partial(h_v a_v)}{\partial u} - \frac{\partial(h_u a_u)}{\partial v}\right)\hat{\mathbf{w}}\end{aligned}$$

- You should show that can be written more compactly as:

Curl in curvi coords is:

$$\mathrm{curl}\mathbf{a}(u,v,w) \;=\; \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{\mathbf{u}} & h_v\hat{\mathbf{v}} & h_w\hat{\mathbf{w}} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u a_u & h_v a_v & h_w a_w \end{vmatrix}$$

The Laplacian in curvilinear coordinates

- Substitute the components of grad$U$ into the expression for div**a** ...
- Much grinding gives the following expression for the Laplacian in general orthogonal co-ordinates:

Laplacian in curvi coords is:

$$\nabla^2 U = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial U}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_w h_u}{h_v}\frac{\partial U}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial U}{\partial w}\right)\right] .$$

Grad, etc, the 3D polar coordinate

- There is no need slavishly to memorize the above derivations or their results.
- More important is to realize why the expressions look suddenly more complicated in curvilinear coordinates
- We are now going to specialize our expressions for the polar family
- As they are 3D entities, we need consider only cylindrical and spherical polars.

Grad, etc, in cylindrical polars

- We recall that $\mathbf{r} = r\cos\theta\hat{\mathbf{\imath}} + r\sin\theta\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}$, and that $h_u = |\partial\mathbf{r}/\partial u|$, and so

$$\begin{aligned} h_r &= \sqrt{(\cos^2\theta + sin^2\theta)} = 1, \\ h_\theta &= \sqrt{(r^2\sin^2\theta + r^2\cos^2\theta)} = r, \\ h_z &= 1 \end{aligned}$$

- Hence, using these and $U(r,\theta,z)$ and $\mathbf{a} = a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} + a_\phi\hat{\boldsymbol{\phi}}$

$$\begin{aligned} \text{grad}U &= \frac{\partial U}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial U}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}} \\ \text{div}\mathbf{a} &= \frac{1}{r}\left(\frac{\partial(ra_r)}{\partial r} + \frac{\partial a_\theta}{\partial\theta}\right) + \frac{\partial a_z}{\partial z} \\ \text{curl}\mathbf{a} &= \left(\frac{1}{r}\frac{\partial a_z}{\partial\theta} - \frac{\partial a_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial(ra_\theta)}{\partial r} - \frac{\partial a_r}{\partial\theta}\right)\hat{\mathbf{k}} \end{aligned}$$

- The derivation of the expression for $\nabla^2 U$ in cylindrical polar co-ordinates is set as a tutorial exercise.

Grad, etc, in spherical polars

- We recall that $\mathbf{r} = r\sin\theta\cos\phi\hat{\mathbf{i}} + r\sin\theta\sin\phi\hat{\mathbf{j}} + r\cos\theta\hat{\mathbf{k}}$ so that

$$\begin{aligned} h_r &= \sqrt{(\sin^2\theta(\cos^2\phi+\sin^2\phi)+\cos^2\theta)} = 1 \\ h_\theta &= \sqrt{(r^2\cos^2\theta(\cos^2\phi+\sin^2\phi)+r^2\sin^2\theta)} = r \\ h_\phi &= \sqrt{(r^2\sin^2\theta(\sin^2\phi+\cos^2\phi)} = r\sin\theta \end{aligned}$$

- Hence

$$\begin{aligned} \text{grad}U &= \frac{\partial U}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\boldsymbol{\phi}} \\ \text{div}\mathbf{a} &= \frac{1}{r^2}\frac{\partial(r^2 a_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(a_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi} \\ \text{curl}\mathbf{a} &= \frac{\hat{\mathbf{r}}}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(a_\phi\sin\theta) - \frac{\partial}{\partial\phi}(a_\theta)\right) + \frac{\hat{\boldsymbol{\theta}}}{r\sin\theta}\left(\frac{\partial}{\partial\phi}(a_r) - \frac{\partial}{\partial r}(a_\phi r\sin\theta)\right) \\ &+ \frac{\hat{\boldsymbol{\phi}}}{r}\left(\frac{\partial}{\partial r}(a_\theta r) - \frac{\partial}{\partial\theta}(a_r)\right) \end{aligned}$$

Question:

Find curl$\mathbf{a}$ in (i) Cartesians and (ii) Spherical polars when $\mathbf{a} = x(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$.

Answer (i):

- In Cartesians, using the pseudo determinant gives

$$\text{curl}\mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & xy & xz \end{vmatrix} = -z\hat{\mathbf{j}} + y\hat{\mathbf{k}}$$

Answer (ii):

- We were told $\mathbf{a} = x(x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}})$.
- In spherical polars $x = r\sin\theta\cos\phi$ and $(x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}) = \mathbf{r}$
- Hence $\mathbf{a} = r\sin\theta\cos\phi\mathbf{r} = r^2\sin\theta\cos\phi\,\hat{\mathbf{r}}$
  or in component form: $a_r = r^2\sin\theta\cos\phi$; $a_\theta = 0$; $a_\phi = 0$.
- Expression for curl (earlier, and HLT):

$$\mathrm{curl}\mathbf{a} = \frac{\hat{\mathbf{r}}}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(a_\phi\sin\theta) - \frac{\partial}{\partial\phi}(a_\theta)\right) + \frac{\hat{\boldsymbol{\theta}}}{r\sin\theta}\left(\frac{\partial}{\partial\phi}(a_r) - \frac{\partial}{\partial r}(a_\phi r\sin\theta)\right) + \frac{\hat{\boldsymbol{\phi}}}{r}\left(\frac{\partial}{\partial r}(a_\theta r) - \frac{\partial}{\partial\theta}(a_r)\right)$$

- Hence

$$\begin{aligned}
\mathrm{curl}\mathbf{a} &= \frac{\hat{\boldsymbol{\theta}}}{r\sin\theta}\left(\frac{\partial}{\partial\phi}(r^2\sin\theta\cos\phi)\right) + \frac{\hat{\boldsymbol{\phi}}}{r}\left(-\frac{\partial}{\partial\theta}(r^2\sin\theta\cos\phi)\right)\\
&= \frac{\hat{\boldsymbol{\theta}}}{r\sin\theta}(-r^2\sin\theta\sin\phi) + \frac{\hat{\boldsymbol{\phi}}}{r}\left(-r^2\cos\theta\cos\phi)\right)\\
&= \hat{\boldsymbol{\theta}}(-r\sin\phi) + \hat{\boldsymbol{\phi}}(-r\cos\theta\cos\phi)
\end{aligned}$$

Example Check: These two results should be the same!

- To check we need $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ in terms of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ ...
- Use $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and

$$\hat{\mathbf{r}} = \frac{1}{h_r}\frac{\partial \mathbf{r}}{\partial r}; \qquad \hat{\boldsymbol{\theta}} = \frac{1}{h_\theta}\frac{\partial \mathbf{r}}{\partial \theta}; \qquad \hat{\boldsymbol{\phi}} = \frac{1}{h_\phi}\frac{\partial \mathbf{r}}{\partial \phi}$$

- Hence, doing the first of these, as $h_r = 1$

$$\hat{\mathbf{r}} = \frac{\partial}{\partial r}(r\sin\theta\cos\phi\hat{\mathbf{i}} + r\sin\theta\sin\phi\hat{\mathbf{j}} + r\cos\theta\hat{\mathbf{j}}) = (\mathrm{s}\theta\mathrm{c}\phi\hat{\mathbf{i}} + \mathrm{s}\theta\mathrm{s}\phi\hat{\mathbf{j}} + \mathrm{c}\theta\hat{\mathbf{j}})$$

- Which gives the top row of the matrix. Grind to find the rest ...

$$\begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix} = [R] \begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix}$$

- Don't be shocked to see a rotation matrix $[R]$! We are rotating one right-handed orthogonal coord system into another.

- Now we convert the spherical polar expression in Cartesians ...

$$
\begin{aligned}
\mathrm{curl}\,\mathbf{a} &= \hat{\boldsymbol{\theta}}(-r\sin\phi) + \hat{\boldsymbol{\phi}}(-r\cos\theta\cos\phi) = -r[0, \sin\phi, \cos\theta\cos\phi]\begin{bmatrix}\hat{\mathbf{r}}\\ \hat{\boldsymbol{\theta}}\\ \hat{\boldsymbol{\phi}}\end{bmatrix} \\
&= -r[0, \sin\phi, \cos\theta\cos\phi]\begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{i}}\\ \hat{\mathbf{j}}\\ \hat{\mathbf{k}}\end{bmatrix} \\
&= -r\sin\phi(\cos\theta\cos\phi\hat{\mathbf{i}} + \cos\theta\sin\phi\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}}) + \\
&\qquad (-r\cos\theta\cos\phi)(-\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}}) \\
&= -r\cos\theta\hat{\mathbf{j}} + r\sin\theta\sin\phi\hat{\mathbf{k}} \\
&= -z\hat{\mathbf{j}} + y\hat{\mathbf{k}}
\end{aligned}
$$

- This is exactly what we got before!

Summary

Take home messages ...

- The key thing when combining operators is to remember that each partial derivative operates on everything to its right.
- The identities (eg in HLT) are not mysterious. They merely provide useful short cuts.
- There is no need slavishly to learn the expressions for grad, div and curl in curvi coords.

  They are in HLT, but

  - you need to know how they originate.
  - you need to be able to hack them out when asked.
- Ditto with the specializations to polars.
- Just as physical vectors are independent of their coordinate systems, so are differential operators.