Vector Algebra and Calculus

1. Revision of vector algebra, scalar product, vector product
2. Triple products, multiple products, applications to geometry
3. Differentiation of vector functions, applications to mechanics
4. **Scalar and vector fields. Line, surface and volume integrals, curvilinear co-ordinates**
5. Vector operators — grad, div and curl
6. Vector Identities, curvilinear co-ordinate systems
7. Gauss' and Stokes' Theorems and extensions
8. Engineering Applications

4. Line, Surface and Volume Integrals I

- We started off
  - being concerned with individual vectors **a**, **b**, **c**, and so on.
- We went on
  - to consider how single vectors vary over time or over some other parameter such as arc length
- In rest of the course, we will be concerned with
  - scalars and vectors which are defined over regions in space
- In this lecture we introduce
  - line, surface and volume integrals
  - definition in *curvilinear coordinates*

Reminder about Scalar and vector fields

If a scalar function $u(\mathbf{r})$ is defined at each $\mathbf{r}$ in some region

- u is a **scalar field** in that region.

Examples: temperature, pressure, altitude, CO_2 concentration

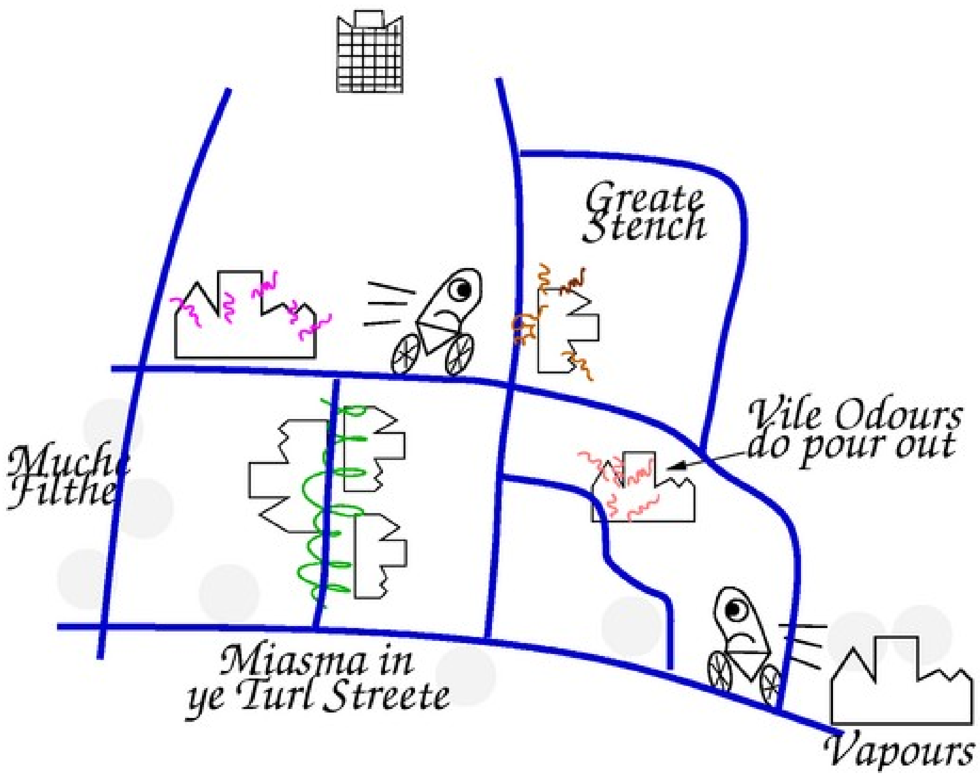
Similarly, if a vector function $\mathbf{v}(\mathbf{r})$ is defined at each point, then

- $\mathbf{v}$ is a **vector field** in that region.

Examples: wind velocity, magnetic field, traffic flows, optical flow, electric fields

In **field theory** our aim is to derive statements about **bulk properties** of scalar and vector fields (rather than to deal with individual scalars or vectors)

- Line integrals are concerned with measuring
  - the integrated interaction with a field as you move through it on some defined path.

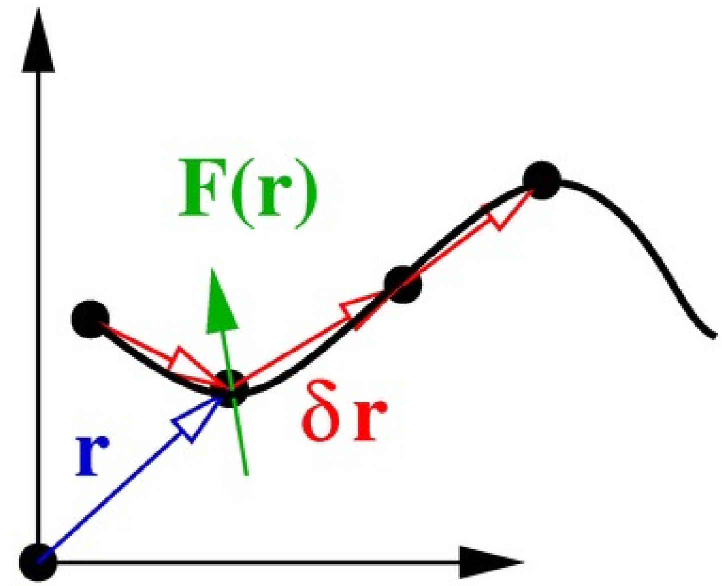


- Eg, given a map showing the pollution density field in Oxford

  how much pollution would you breath in when cycling from college to the Department on different routes?

Vector line integrals

Path L chopped into *vector* segments $\delta\mathbf{r}_i$.
Each segment is multiplied by the field value at that point in space,
Products are summed.
Three types



1: Integrand $U(\mathbf{r})$ is a scalar field. Integral is a vector.

$$\mathbf{I} = \int_L U(\mathbf{r}) d\mathbf{r}$$

2: Integrand $\mathbf{a}(\mathbf{r})$ is a vector field dotted with $d\mathbf{r}$. Integral is a scalar:

$$I = \int_L \mathbf{a}(\mathbf{r}) \cdot d\mathbf{r}$$

3: Integrand $\mathbf{a}(\mathbf{r})$ is a vector field crossed with $d\mathbf{r}$. Integral is vector.

$$\mathbf{I} = \int_L \mathbf{a}(\mathbf{r}) \times d\mathbf{r}$$

♣ Examples

- Total work done by force $\mathbf{F}$ as it moves point from A to B along path C.

  Infinitessimal work done is $dW = \mathbf{F}.d\mathbf{r}$, hence total work is

$$W_C = \int_C \mathbf{F}.d\mathbf{r}$$

- Ampère's law relating magnetic field $\mathbf{B}$ to linked current can be written as

$$\oint_C \mathbf{B}.d\mathbf{r} = \mu_0 I$$

- Force on an element of wire carrying current I, placed in a magnetic field of strength $\mathbf{B}$, is $d\mathbf{F} = I\,d\mathbf{r} \times \mathbf{B}$.

  So total force on loop of wire C :

$$\mathbf{F} = I \oint_C d\mathbf{r} \times \mathbf{B}$$

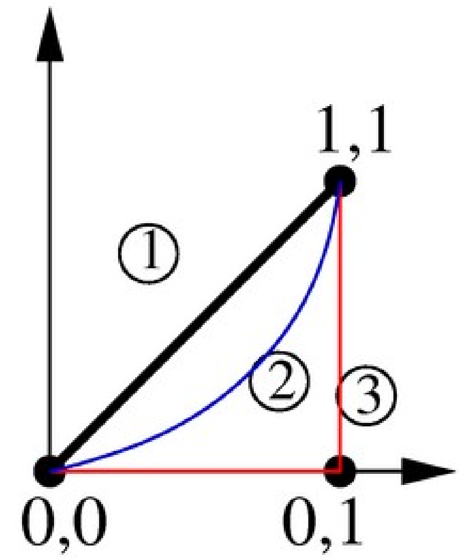
Note: expressions above are beautifully compact in vector notation, and are all independent of coordinate system

♣ Examples

Question: A force $\mathbf{F} = x^2 y\hat{\mathbf{\imath}} + xy^2\hat{\mathbf{\jmath}}$ acts on a body at it moves between $(0,0)$ and $(1,1)$.

Find work done when path is

1. along the line $y = x$.
2. along the curve $y = x^n$.
3. along the x axis to the point $(1,0)$ and then along the line $x = 1$



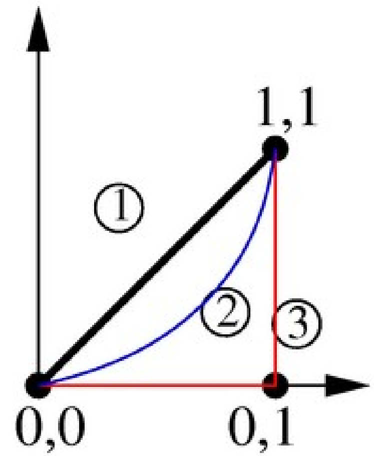
Answer:

In planar Cartesians $\quad d\mathbf{r} = \hat{\mathbf{\imath}}dx + \hat{\mathbf{\jmath}}dy$

Then the work done is

$$\int_L \mathbf{F} \cdot d\mathbf{r} = \int_L (x^2 y\hat{\mathbf{\imath}} + xy^2\hat{\mathbf{\jmath}}) \cdot (\hat{\mathbf{\imath}}dx + \hat{\mathbf{\jmath}}dy) = \int_L (x^2 y\,dx + xy^2 dy)\ .$$

Example Path 1

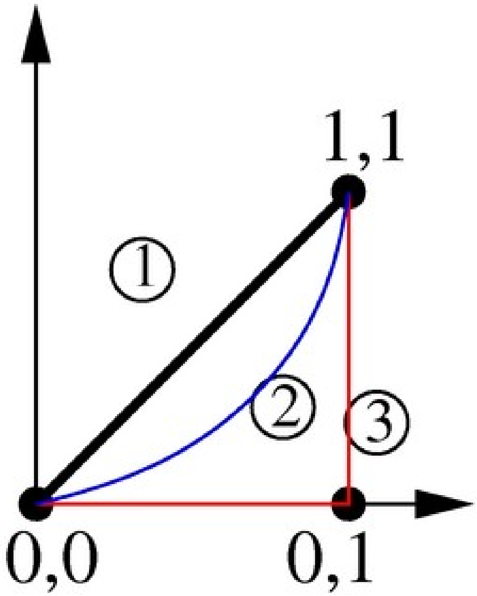


PATH 1: For the path $y = x$ we find that $dy = dx$. So it is easiest to convert all y references to x.

$$\begin{aligned}\int_{(0,0)}^{(1,1)} (x^2 y\,dx + xy^2\,dy) &= \int_{x=0}^{x=1} (x^2 x\,dx + x x^2\,dx) \\ &= \int_{x=0}^{x=1} 2x^3\,dx \\ &= \left[x^4/2\right|_{x=0}^{x=1} = 1/2\ .\end{aligned}$$

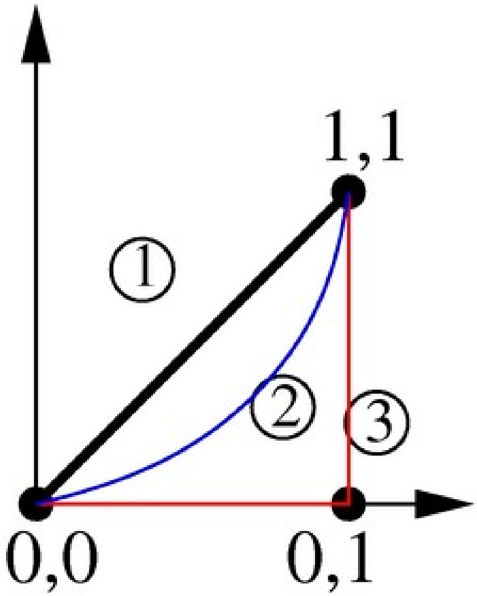
NB! Although x, y involved these are NOT double integrals. Why not?

Example Path 2



PATH 2: For path $y = x^n$ find $dy = nx^{n-1}dx$
Again convert y references to x.

$$\begin{aligned}
\int_{(0,0)}^{(1,1)} (x^2y\,dx + xy^2dy) &= \int_{x=0}^{x=1} (x^{n+2}dx + nx^{n-1}.x.x^{2n}dx) \\
&= \int_{x=0}^{x=1} (x^{n+2}dx + nx^{3n}dx) \\
&= \frac{1}{n+3} + \frac{n}{3n+1}
\end{aligned}$$



PATH 3: not smooth, so **break into two**. Along the first section, $y = 0$ and $dy = 0$, along second section $x = 1$ and $dx = 0$:

$$\begin{aligned}\int_A^B (x^2y\,dx + xy^2dy) &= \int_{x=0}^{x=1}(x^2 0\,dx) + \int_{y=0}^{y=1} 1.y^2dy \\ &= 0 + \left[y^3/3\right|_{y=0}^{y=1} \\ &= 1/3\ .\end{aligned}$$

Line integral depends on path taken

♣ Example 2

Question 2: Repeat path (2), but now using the Force $\mathbf{F} = xy^2\hat{\mathbf{\imath}} + x^2y\hat{\mathbf{\jmath}}$.

Answer 2:

$$\mathbf{F}\cdot(\hat{\mathbf{\imath}}dx + \hat{\mathbf{\jmath}}dy) = xy^2\ dx \quad + \quad x^2y\ dy.$$

For the path $y = x^n$ we find that $dy = nx^{n-1}dx$, so

$$\begin{aligned}\int_{(0,0)}^{(1,1)}(xy^2dx + x^2ydy) &= \int_{x=0}^{x=1}(x^{2n+1}dx + nx^{n-1}x^2x^ndx)\\ &= \int_{x=0}^{x=1}(x^{2n+1}dx + nx^{2n+1}dx)\\ &= \frac{1}{2n+2} + \frac{n}{2n+2} \quad = \quad \frac{1}{2}\end{aligned}$$

This is independent of n, so

This line is independent of path!

Can we understand why?

Line integrals in Conservative fields

- Write

$$g(x, y) = x^2y^2/2$$

- Then the perfect differential is

$$\begin{aligned} dg &= \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy \\ &= y^2x dx + x^2y dy \end{aligned}$$

- So our line integral

$$\int \mathbf{F} \cdot d\mathbf{r} = \int_A^B (y^2x dx + yx^2 dy) = \int_A^B dg = g_B - g_A$$

- It depends solely on the value of g at the start and end points, and not at all on the path
- A vector field which gives rise to line integrals which are independent of paths is called

a conservative field

Some questions about conservative fields

One sort of line integral performs the integration around a complete loop. It is denoted $\oint$

1. If $\mathbf{E}$ is a conservative field, what is the value of $\oint \mathbf{E} \cdot d\mathbf{r}$?
2. If $\mathbf{E}_1$ and $\mathbf{E}_2$ is conservative, is $\mathbf{E}_1 + \mathbf{E}_2$ conservative?
3. Later we will show that the electric field around a point charge q

$$\mathbf{E} = Kq\frac{\hat{\mathbf{r}}}{r^2} \qquad K = 1/4\pi\epsilon_r\epsilon_0$$

   is conservative. Are all electric fields conservative?

4. If $\mathbf{E}$ is the electric field, the potential function is

$$\phi = -\int \mathbf{E} \cdot d\mathbf{r} \ .$$

   So are all electric fields conservative?

$$I = \int_L F(x, y, z) ds \quad ,$$

where path L along a curve is defined as $x = x(p)$, $y = y(p)$, $z = z(p)$

- These integrals don't appear to involve vectors, but they could be reformulated to!
- First, convert the function to $F(p)$, writing

  $$I = \int_{p_{\text{start}}}^{p_{\text{end}}} F(p) \frac{ds}{dp}\, dp$$

  where (Lecture 3)

  $$\frac{ds}{dp} = \left[\left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2 + \left(\frac{dz}{dp}\right)^2 \right]^{1/2} \quad .$$

- Then do the (now straightforward) integral w.r.t. p.

Line integrals involving scalar arc-length: Special cases

$$I = \int_L F(x, y, z) ds$$

1: If the parameter is arc-length s and the path L is $x = x(s)$, $y = y(s)$, $z = z(s)$.

Convert the function to $F(s)$, writing

$$I = \int_{s_{\text{start}}}^{s_{\text{end}}} F(s)\, ds$$

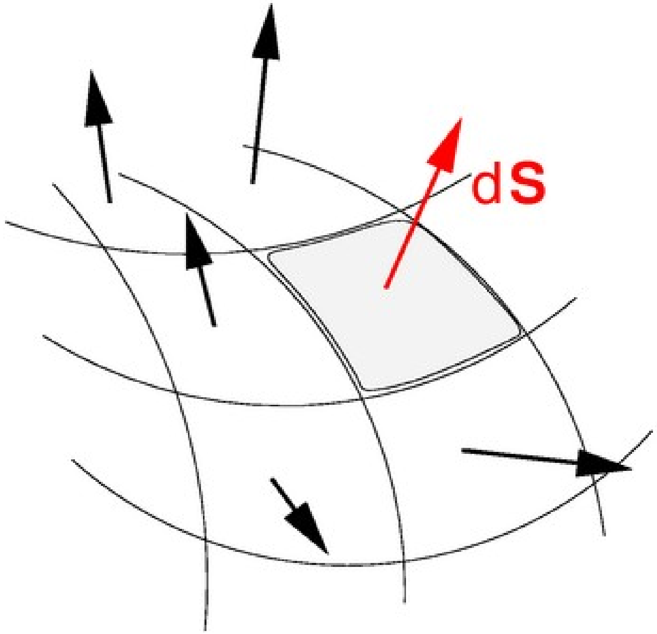
2: If p is x — so $y = y(x)$ and $z = z(x)$ (or similar for $p = y$ or $p = z$)

$$I = \int_{x_{\text{start}}}^{x_{\text{end}}} F(x) \left[1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right]^{1/2} dx \quad .$$

Surface integrals

Surface S is divided into infinitesimal vector elements of area $d\mathbf{S}$:

- the dirn of the vector $d\mathbf{S}$ is the surface normal
- its magnitude represents the area of the element.



Again there are three possibilities:

1: $\int_S U d\mathbf{S}$ — scalar field U; vector integral.

2: $\int_S \mathbf{a} \cdot d\mathbf{S}$ — vector field $\mathbf{a}$; scalar integral.

3: $\int_S \mathbf{a} \times d\mathbf{S}$ — vector field $\mathbf{a}$; vector integral.

Physical example of surface integral

- Physical examples of surface integrals often involve the idea of **flux** of a vector field through a surface

$$\int_S \mathbf{a} \cdot d\mathbf{S}$$

- Mass of fluid crossing a surface element $d\mathbf{S}$ at $\mathbf{r}$ in time dt is

$$dM = \rho \, \mathbf{v} \cdot d\mathbf{S} dt$$

  Total rate of gain of mass can be expressed as a surface integral:

$$\frac{dM}{dt} = \int_S \rho(\mathbf{r}) \mathbf{v}(\mathbf{r}) \cdot d\mathbf{S}$$

- Note again that expression is coordinate free.

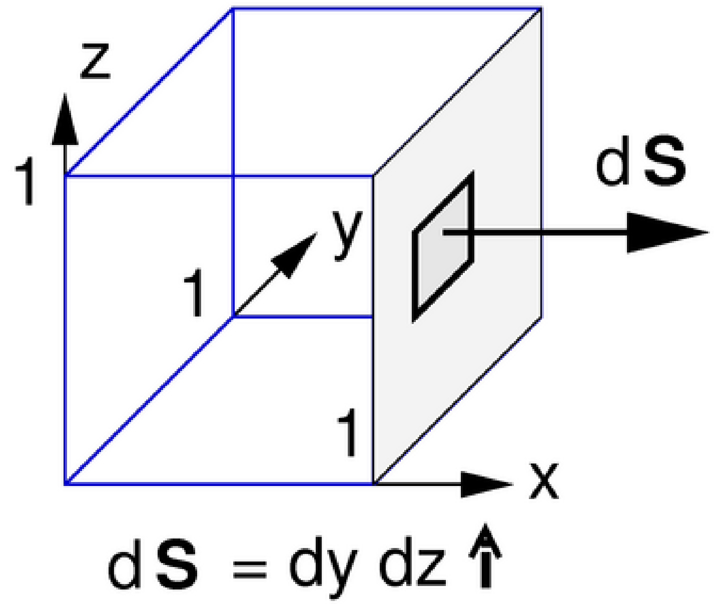
♣ Example

Question: Evaluate $\int \mathbf{F} \cdot d\mathbf{S}$ over the $x = 1$ side of the cube shown in the figure when $\mathbf{F} = y\hat{\mathbf{\imath}} + z\hat{\mathbf{\jmath}} + x\hat{\mathbf{k}}$.

Answer: $d\mathbf{S}$ is perp to the surface. Often, the surface will enclose a volume, and the surface direction is everywhere out of the volume
For the $x = 1$ face of the cube

$$d\mathbf{S} = dydz\hat{\mathbf{\imath}}$$

$$\begin{aligned}\int_S \mathbf{F} \cdot d\mathbf{S} &= \int\int (y\hat{\mathbf{\imath}} + z\hat{\mathbf{\jmath}} + x\hat{\mathbf{k}}) \cdot dydz\hat{\mathbf{\imath}} \\ &= \int_{y=0}^{y=1}\int_{z=0}^{z=1} ydydz \\ &= \frac{1}{2}y^2\Big|_0^1 z\Big|_0^1 \\ &= \frac{1}{2} \quad .\end{aligned}$$



Volume integrals

- The definition of the volume integral is again taken as the limit of a sum of products as the size of the volume element tends to zero.
- One obvious difference though is that the element of volume is a scalar.
- The possibilities are:

1: $\int_V U(\mathbf{r})dV$ — scalar field; scalar integral (1P1 stuff!)

2: $\int_V \mathbf{a}(\mathbf{r})dV$ — vector field; vector integral. In this case one can treat each component separately.

$$\begin{aligned}\int_V \mathbf{a}dV &= \int_V a_1(x,y,z)\hat{\mathbf{i}}dV + \int_V a_2(x,y,z)\hat{\mathbf{j}}dV + \int_V a_3(x,y,z)\hat{\mathbf{k}}dV \\ &= \hat{\mathbf{i}}\int_V a_1(x,y,z)dV + \hat{\mathbf{j}}\int_V a_2(x,y,z)dV + \hat{\mathbf{k}}\int_V a_3(x,y,z)dV\end{aligned}$$

So, $3\times$ 1P1 stuff.

Changing variables: curvilinear coordinates

- Before dealing with further examples of line, surface and volume integrals ...

  ... it is important to understand how to convert an integral from one set of coordinates into another more suited to the geometry or symmetry of the problem.

- You saw how to do this for scalar volume integrals in 1P1 (and we've seen that volume integrals can always be handled as scalars) ...

  ... but we need to understand where Jacobians came from, and how we can apply the mechanism more generally.

- You may find the general problem slightly heavy going ...

  ... the good news is that we will deal with some special cases first that should be at least vaguely familiar

Changing variables: curvilinear coordinates

- A line integral in Cartesian coordinates used

$$\mathbf{r} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}} \qquad \text{and} \qquad d\mathbf{r} = dx\hat{\mathbf{\imath}} + dy\hat{\mathbf{\jmath}} + dz\hat{\mathbf{k}}$$

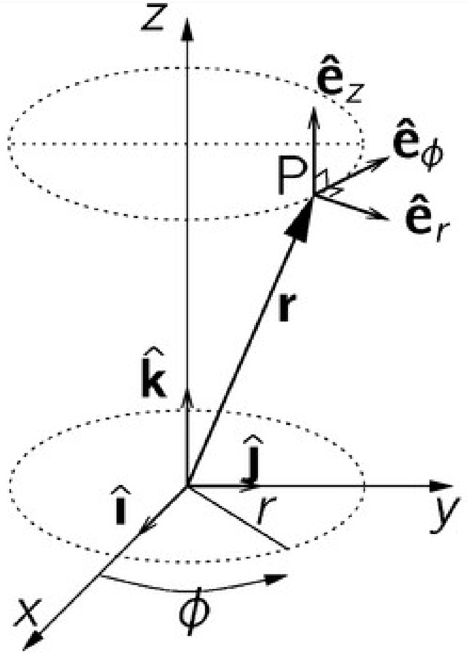
- You can be sure that length scales are properly handled because

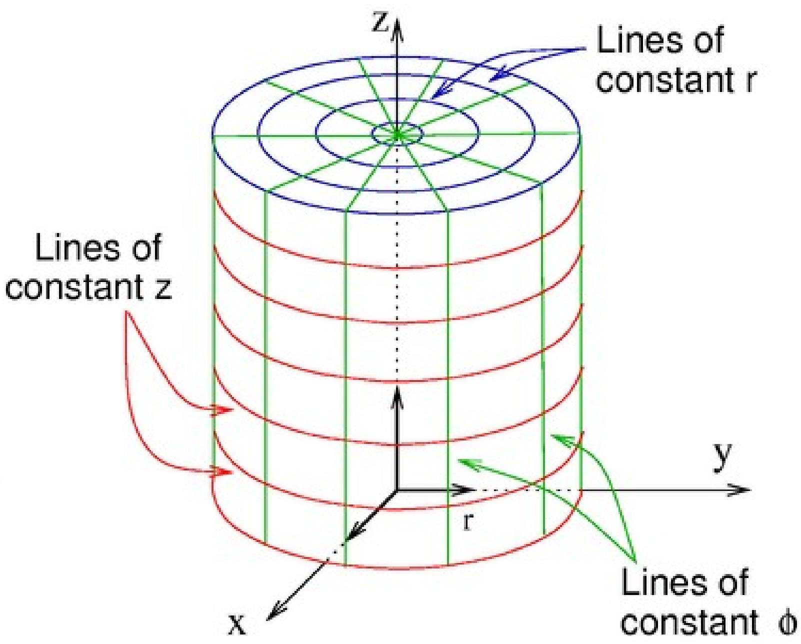
$$|d\mathbf{r}| = ds = \sqrt{dx^2 + dy^2 + dz^2}.$$

- But often symmetry screams at you to use another coordinate system:
  - likely to be plane, cylindrical, or spherical polars,
  - but can be something more exotic like "u, v, w"
- A "u, v, w", "r, ϕ, θ" system is a **curvilinear coordinate system**
- **But here's the bad news: Length scales are screwed up**

$$\begin{aligned} \mathbf{r} &\neq u\hat{\mathbf{u}} + v\hat{\mathbf{v}} + w\hat{\mathbf{w}} \\ d\mathbf{r} &\neq du\hat{\mathbf{u}} + dv\hat{\mathbf{v}} + dw\hat{\mathbf{w}} \\ |d\mathbf{r}| = ds &\neq \sqrt{du^2 + dv^2 + dw^2}\ . \end{aligned}$$

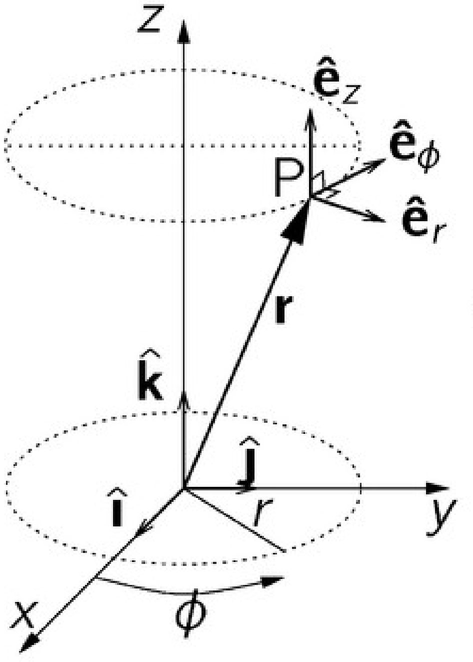
Cylindrical polar coordinates

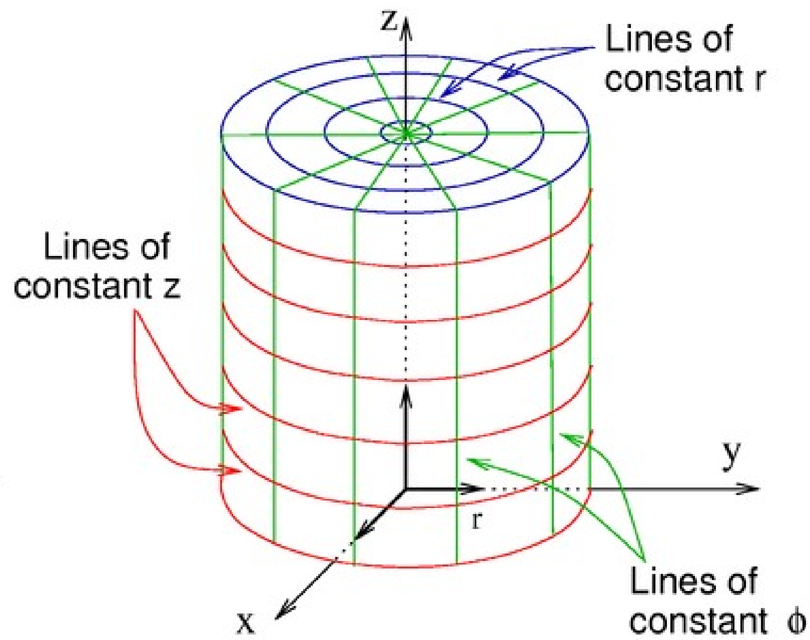




$$\mathbf{r} \;=\; x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = r\cos\phi\hat{\mathbf{i}} + r\sin\phi\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

$\frac{\partial \mathbf{r}}{\partial r}$ represents direction in which (instantaneously) r changing while other two coords stay const. It is **tangent** to surfaces of constant ϕ and z





$$\mathbf{r} = r\cos\phi\hat{\mathbf{i}} + r\sin\phi\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

$$\mathbf{e}_r = \frac{\partial\mathbf{r}}{\partial r} = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}}$$
$$\mathbf{e}_\phi = \frac{\partial\mathbf{r}}{\partial\phi} = -r\sin\phi\hat{\mathbf{i}} + r\cos\phi\hat{\mathbf{j}}$$
$$\mathbf{e}_z = \frac{\partial\mathbf{r}}{\partial z} = \hat{\mathbf{k}}$$

form a basis set for infinitessimal vector displacements in the position of P, $d\mathbf{r}$.

- More usual to normalise the basis vectors

$$\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_z$$

$$
\begin{aligned}
d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial r} dr + \frac{\partial \mathbf{r}}{\partial \phi} d\phi + \frac{\partial \mathbf{r}}{\partial z} dz \\
&= dr\mathbf{e}_r + d\phi\mathbf{e}_\phi + dz\mathbf{e}_z \\
&= dr\hat{\mathbf{e}}_r + rd\phi\hat{\mathbf{e}}_\phi + dz\hat{\mathbf{e}}_z
\end{aligned}
$$

- **NOTE:** In cylindrical polars, small change $d\phi$ keeping r and z constant results in displacement of

$$ds = |d\mathbf{r}| = \sqrt{r^2(d\phi)^2} = rd\phi$$

  THUS: size of (infinitessimal) displacement depends on value of r

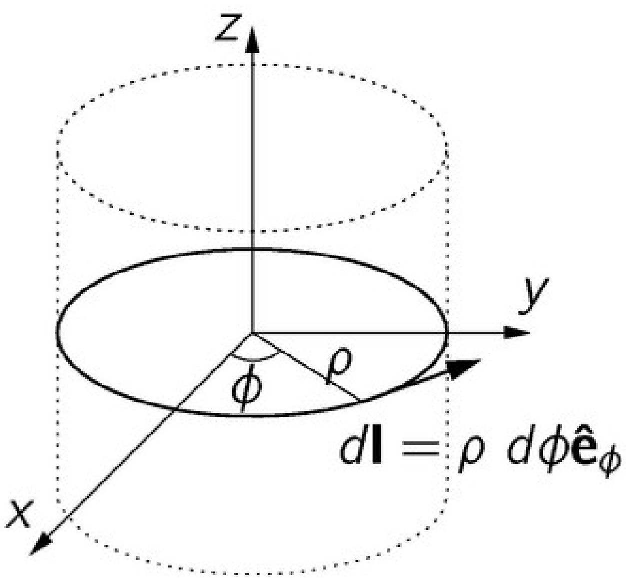
- r is **scale factor** or **metric coefficient**

- In cylindrical polars scale factors are 1, r and 1.

♣ Example: line integral in cylindrical polars

Question: Evaluate $\oint_C \mathbf{a}\cdot d\mathbf{l}$, where $\mathbf{a} = x^3\hat{\mathbf{j}} - y^3\hat{\mathbf{i}} + x^2y\hat{\mathbf{k}}$ and C is the circle of radius ρ in the $z = 0$ plane, centred on the origin.

Answer:



$$\mathbf{a} = \rho^3(-\sin^3\phi\hat{\mathbf{i}} + \cos^3\phi\hat{\mathbf{j}} + \cos^2\phi\sin\phi\hat{\mathbf{k}})$$

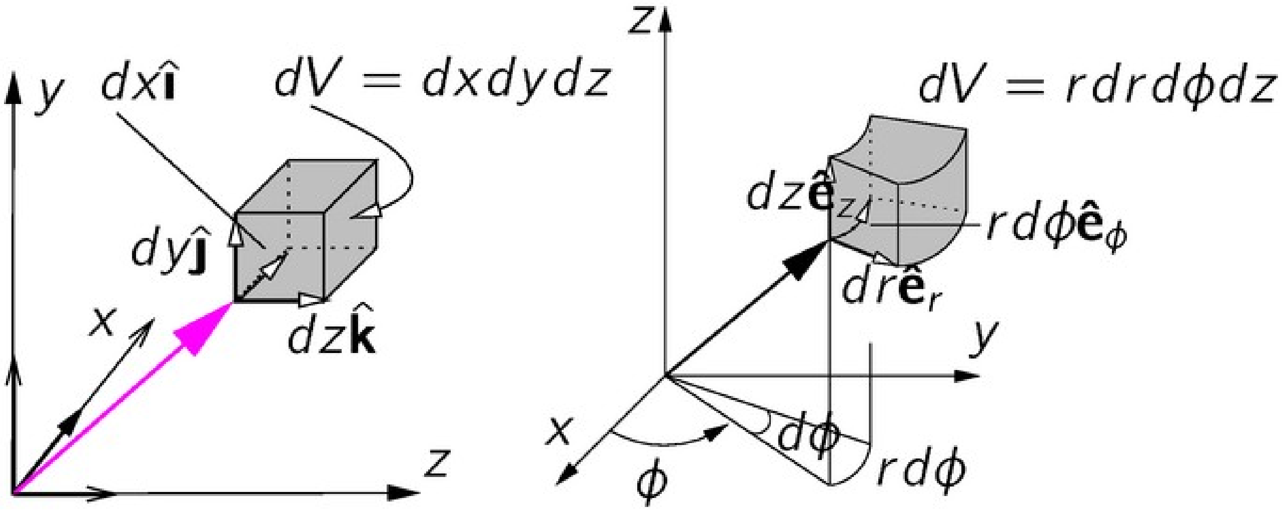
and (since $dz = dr = 0$ on the path)

$$\begin{aligned} d\mathbf{l} &= \rho\ d\phi\ \hat{\mathbf{e}}_\phi \\ &= \rho d\phi(-\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}}) \end{aligned}$$

so that

$$\oint_C \mathbf{a}\cdot d\mathbf{l} = \int_0^{2\pi} \rho^4(\sin^4\phi + \cos^4\phi)d\phi = \frac{3\pi}{2}\rho^4$$

Volume integrals in cylindrical polars



In Cartesians, volume element given by

$$dV = dx\hat{\mathbf{i}}.(dy\hat{\mathbf{j}} \times dz\hat{\mathbf{k}}) = dxdydz$$

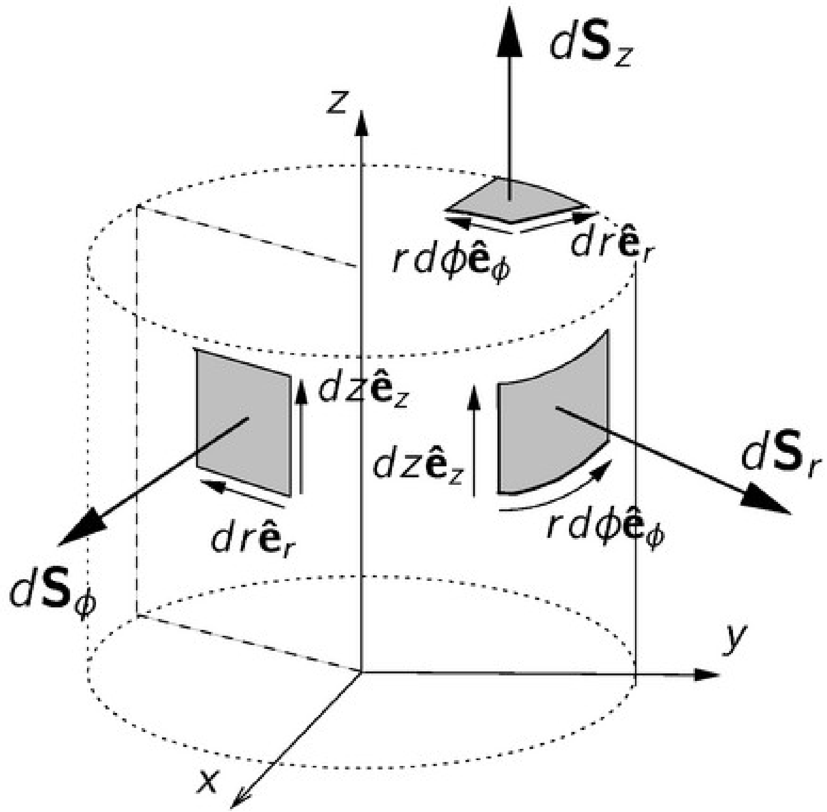
In cylindrical polars, volume element given by

$$dV = dr\hat{\mathbf{e}}_r.(rd\phi\hat{\mathbf{e}}_\phi \times dz\hat{\mathbf{e}}_z) = rd\phi drdz$$

Note: Volume is scalar triple product, hence can be written as a determinant:

$$dV = \begin{vmatrix} \hat{\mathbf{e}}_r dr \\ \hat{\mathbf{e}}_\phi r d\phi \\ \hat{\mathbf{e}}_z dz \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} drd\phi dz$$

Surface integrals in cylindrical polars



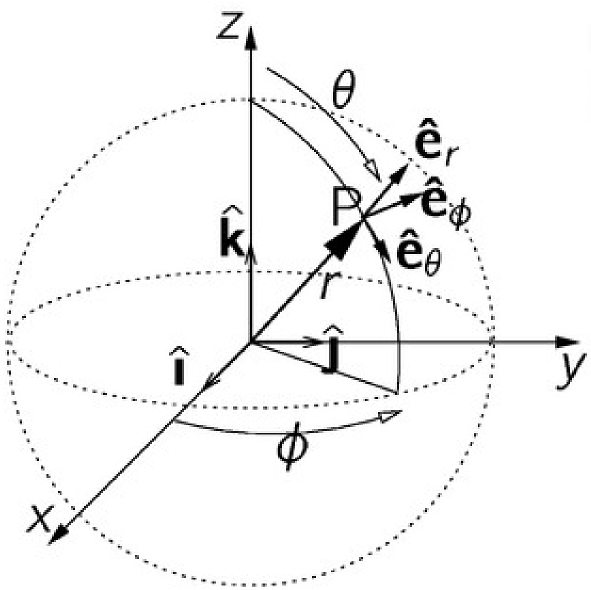
In Cartesians, for surface element with normal $\hat{\mathbf{\imath}}$ we have

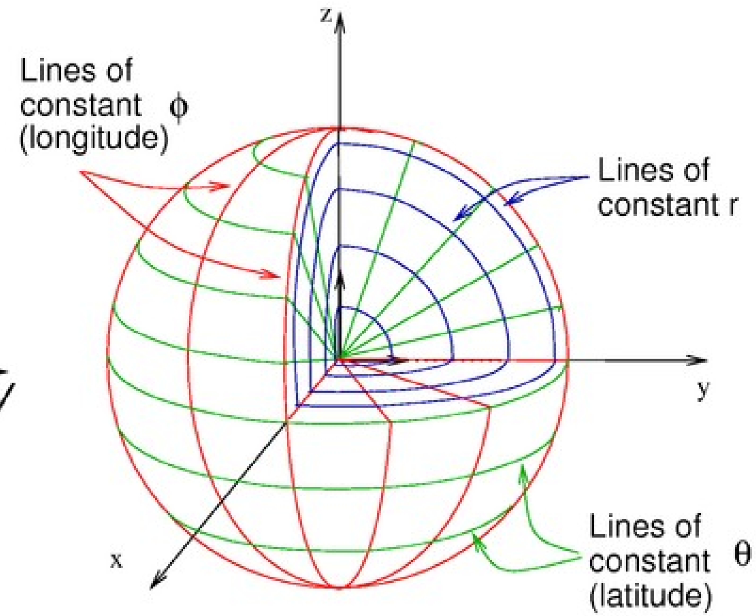
$$d\mathbf{S} \;=\; dy\hat{\mathbf{\jmath}} \times dz\hat{\mathbf{k}} = dydz\hat{\mathbf{\imath}} = \hat{\mathbf{\imath}}dS$$

Cylindrical polars: surface area elements given by:

$$d\mathbf{S}_z \;=\; dr\hat{\mathbf{e}}_r \times rd\phi\hat{\mathbf{e}}_\phi = rdrd\phi\hat{\mathbf{e}}_z$$
$$d\mathbf{S}_r \;=\; rd\phi\hat{\mathbf{e}}_\phi \times dz\hat{\mathbf{e}}_z = rd\phi dz\hat{\mathbf{e}}_r$$

Spherical polars





$$\mathbf{r} \;=\; x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}} = r\sin\theta\cos\phi\hat{\mathbf{\imath}} + r\sin\theta\sin\phi\hat{\mathbf{\jmath}} + r\cos\theta\hat{\mathbf{k}}$$

$$\begin{aligned}
\mathbf{e}_r = \frac{\partial\mathbf{r}}{\partial r} &= \sin\theta\cos\phi\hat{\mathbf{\imath}} + \sin\theta\sin\phi\hat{\mathbf{\jmath}} + \cos\theta\hat{\mathbf{k}} &= \hat{\mathbf{e}}_r \\
\mathbf{e}_\theta = \frac{\partial\mathbf{r}}{\partial\theta} &= r\cos\theta\cos\phi\hat{\mathbf{\imath}} + r\cos\theta\sin\phi\hat{\mathbf{\jmath}} - r\sin\theta\hat{\mathbf{k}} &= r\hat{\mathbf{e}}_\theta \\
\mathbf{e}_\phi = \frac{\partial\mathbf{r}}{\partial\phi} &= -r\sin\theta\sin\phi\hat{\mathbf{\imath}} + r\sin\theta\cos\phi\hat{\mathbf{\jmath}} &= r\sin\theta\hat{\mathbf{e}}_\phi
\end{aligned}$$

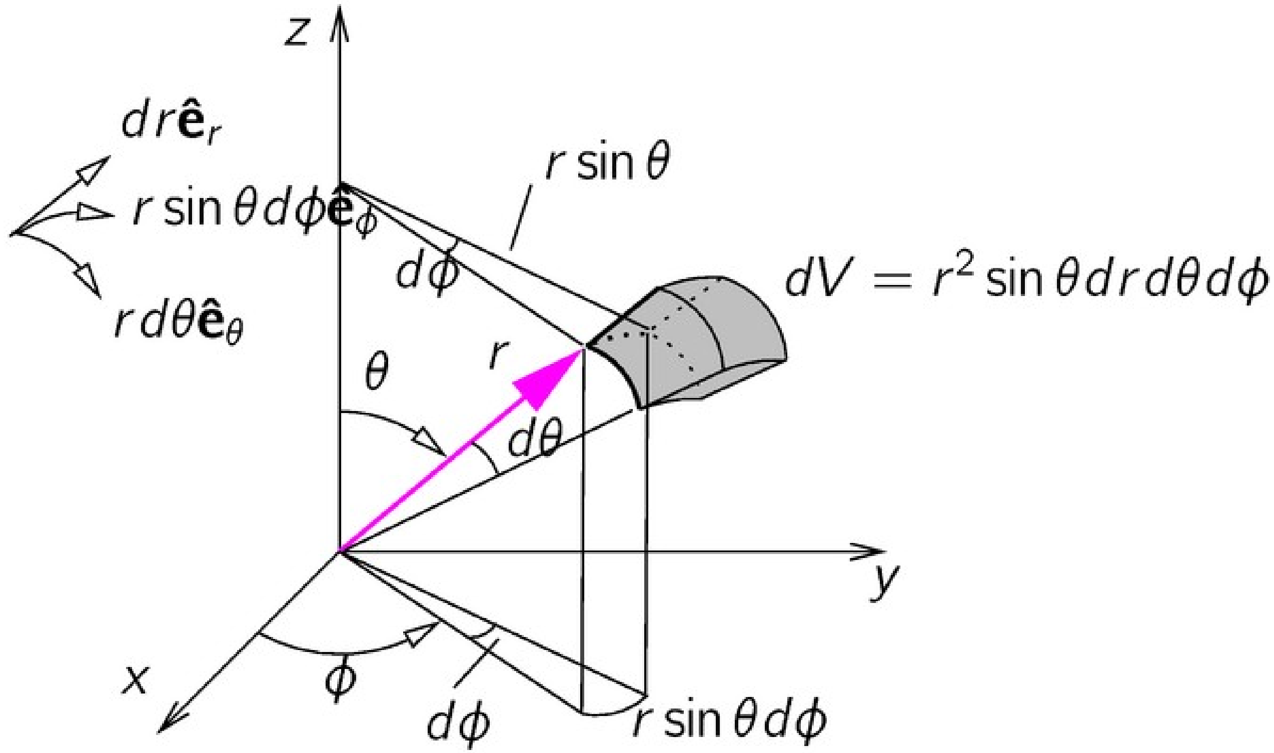
$$
\begin{aligned}
\mathbf{e}_r = \frac{\partial \mathbf{r}}{\partial r} &= \sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}} &= \hat{\mathbf{e}}_r \\
\mathbf{e}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} &= r\cos\theta\cos\phi\hat{\mathbf{i}} + r\cos\theta\sin\phi\hat{\mathbf{j}} - r\sin\theta\hat{\mathbf{k}} &= r\hat{\mathbf{e}}_\theta \\
\mathbf{e}_\phi = \frac{\partial \mathbf{r}}{\partial \phi} &= -r\sin\theta\sin\phi\hat{\mathbf{i}} + r\sin\theta\cos\phi\hat{\mathbf{j}} &= r\sin\theta\hat{\mathbf{e}}_\phi
\end{aligned}
$$

- Small displacement $d\mathbf{r}$ given by:

$$
\begin{aligned}
d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial r}dr + \frac{\partial \mathbf{r}}{\partial \theta}d\theta + \frac{\partial \mathbf{r}}{\partial \phi}d\phi \\
&= dr\mathbf{e}_r + d\theta\mathbf{e}_\theta + d\phi\mathbf{e}_\phi \\
&= dr\hat{\mathbf{e}}_r + r\,d\theta\hat{\mathbf{e}}_\theta + r\sin\theta\,d\phi\hat{\mathbf{e}}_\phi
\end{aligned}
$$

- Thus, metric coefficients are 1, r, $r\sin\theta$.

Volume integrals in spherical polars



- Volume element given by

$$dV = dr\hat{\mathbf{e}}_r.(r d\theta\hat{\mathbf{e}}_\theta \times r\sin\theta d\phi\hat{\mathbf{e}}_\phi) = r^2\sin\theta dr d\theta d\phi$$

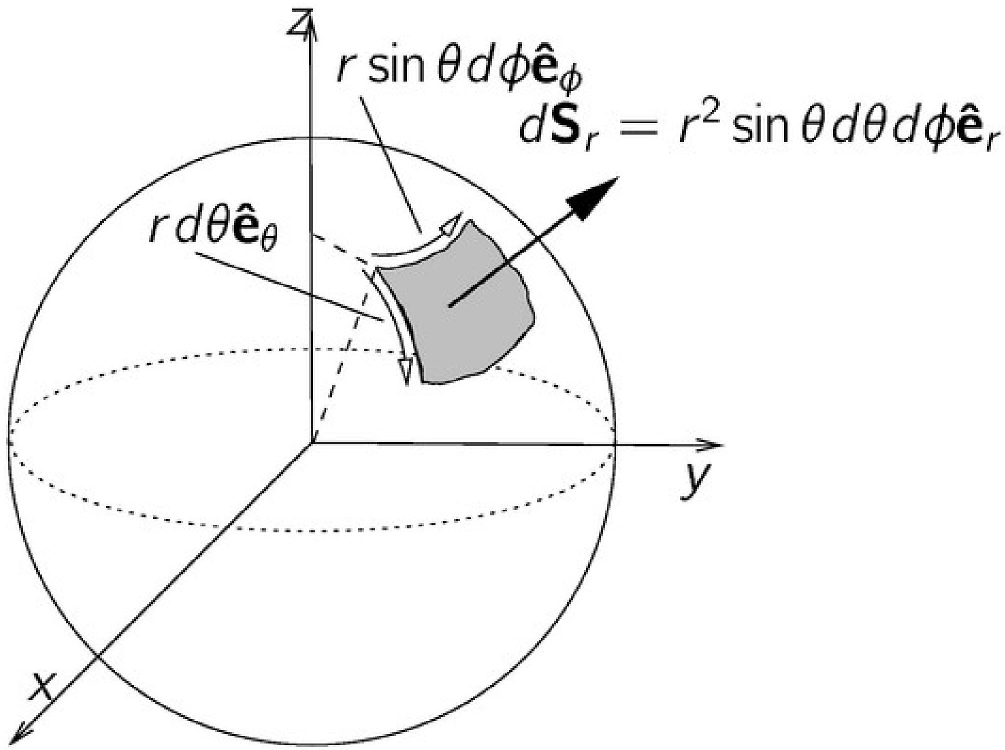
- Note again that this volume could be written as a determinant

Surface integrals in spherical polars

Three possibilities, but most useful are surfaces of constant r

The surface element $d\mathbf{S}_r$ is given by

$$\begin{aligned} d\mathbf{S}_r &= r\,d\theta\,\hat{\mathbf{e}}_\theta \times r\sin\theta\,d\phi\,\hat{\mathbf{e}}_\phi \\ &= r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{e}}_r \end{aligned}$$



♣ Example: Surface integral in spherical polars

Q Evaluate $\int_S \mathbf{a} \cdot d\mathbf{S}$, where $\mathbf{a} = z^3\hat{\mathbf{k}}$ and S is the sphere of radius A centred on the origin.

A In general:

$$z = r\cos\theta \qquad d\mathbf{S} = r^2 \sin\theta d\theta d\phi \hat{\mathbf{e}}_r$$

On surface of the sphere, $r = A$, so that

$$\mathbf{a} = A^3 cos^3\theta\hat{\mathbf{k}} \qquad d\mathbf{S} = A^2 \sin\theta \; d\theta \; d\phi\hat{\mathbf{e}}_r$$

Hence

$$\begin{aligned}
\int_S \mathbf{a} \cdot d\mathbf{S} &= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} A^3 cos^3\theta \; A^2 \sin\theta \; [\hat{\mathbf{e}}_r \cdot \hat{\mathbf{k}}] \; d\theta d\phi \\
&= A^5 \int_0^{2\pi} d\phi \; \int_0^{\pi} cos^3\theta \sin\theta [\cos\theta] \; d\theta \\
&= 2\pi A^5 \frac{1}{5}\left[-\cos^5\theta\right]_0^{\pi} = \frac{4\pi A^5}{5}
\end{aligned}$$

General curvilinear coordinates

Suppose

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w)$$

So

$$\mathbf{r} = x(u, v, w)\hat{\mathbf{\imath}} + y(u, v, w)\hat{\mathbf{\jmath}} + z(u, v, w)\hat{\mathbf{k}}$$

and

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\hat{\mathbf{\imath}} + \frac{\partial y}{\partial u}\hat{\mathbf{\jmath}} + \frac{\partial z}{\partial u}\hat{\mathbf{k}}$$

[similarly for partials with respect to v and w]

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}du + \frac{\partial \mathbf{r}}{\partial v}dv + \frac{\partial \mathbf{r}}{\partial w}dw$$

Postion vector:

$$\mathbf{r} = x(u,v,w)\hat{\mathbf{\imath}} + y(u,v,w)\hat{\mathbf{\jmath}} + z(u,v,w)\hat{\mathbf{k}}$$

Infinitessimal displacement:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}du + \frac{\partial \mathbf{r}}{\partial v}dv + \frac{\partial \mathbf{r}}{\partial w}dw$$

Local basis:

$$\begin{aligned}
\mathbf{e}_u &= \frac{\partial \mathbf{r}}{\partial u} = \left|\frac{\partial \mathbf{r}}{\partial u}\right| \hat{\mathbf{e}}_u = h_u \hat{\mathbf{e}}_u \\
\mathbf{e}_v &= \frac{\partial \mathbf{r}}{\partial v} = \left|\frac{\partial \mathbf{r}}{\partial v}\right| \hat{\mathbf{e}}_v = h_v \hat{\mathbf{e}}_v \\
\mathbf{e}_w &= \frac{\partial \mathbf{r}}{\partial w} = \left|\frac{\partial \mathbf{r}}{\partial w}\right| \hat{\mathbf{e}}_w = h_w \hat{\mathbf{e}}_w
\end{aligned}$$

- **Metric coefficients**

$$h_u = |\frac{\partial \mathbf{r}}{\partial u}|,\, h_v = |\frac{\partial \mathbf{r}}{\partial v}| \quad \text{and} \quad h_w = |\frac{\partial \mathbf{r}}{\partial w}|$$

- **Volume element**

$$dV = h_u du \hat{\mathbf{e}}_u . (h_v dv \hat{\mathbf{e}}_v \times h_w dw \hat{\mathbf{e}}_w)$$

and simplifies *if the coordinate system is orthonormal* (since $\hat{\mathbf{e}}_u.(\hat{\mathbf{e}}_v \times \hat{\mathbf{e}}_w) = 1$) to

$$dV = h_u h_v h_w du dv dw$$

- **Surface element** (normal to constant w, say)

$$d\mathbf{S} = h_u du \hat{\mathbf{e}}_u \times h_v dv \hat{\mathbf{e}}_v$$

and simplifies *if the coordinate system is orthogonal* to

$$d\mathbf{S} = h_u h_v du dv \hat{\mathbf{e}}_w$$

Summary

- We introduced line, surface and volume integrals involving vector fields.
- We defined *curvilinear coordinates*, and realized that metric coefficient were necessary to relate change in an arbitrary coordinate to a length scale.
- We showed in detail how line, surface and volume elements are derived, and how the results specialized for orthogonal curvilinear system, in particular plane, cylindrical and spherical polar coordinates.
- The origin of the Jacobian was clarified.