Vector Algebra and Calculus

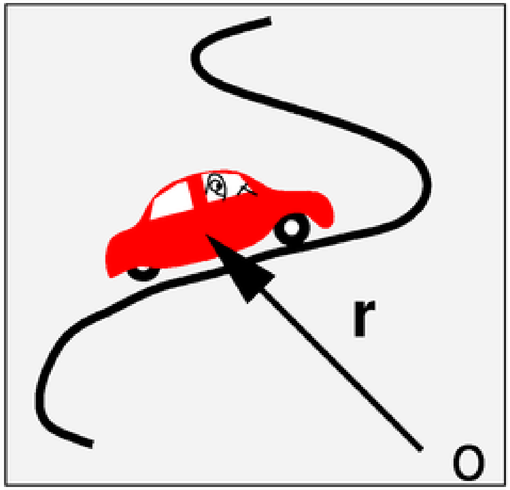
1. Revision of vector algebra, scalar product, vector product
2. Triple products, multiple products, applications to geometry
3. **Differentiation of vector functions, applications to mechanics**
4. Scalar and vector fields. Line, surface and volume integrals, curvilinear co-ordinates
5. Vector operators — grad, div and curl
6. Vector Identities, curvilinear co-ordinate systems
7. Gauss' and Stokes' Theorems and extensions
8. Engineering Applications

3. Differentiating Vector Functions of a Single Variable

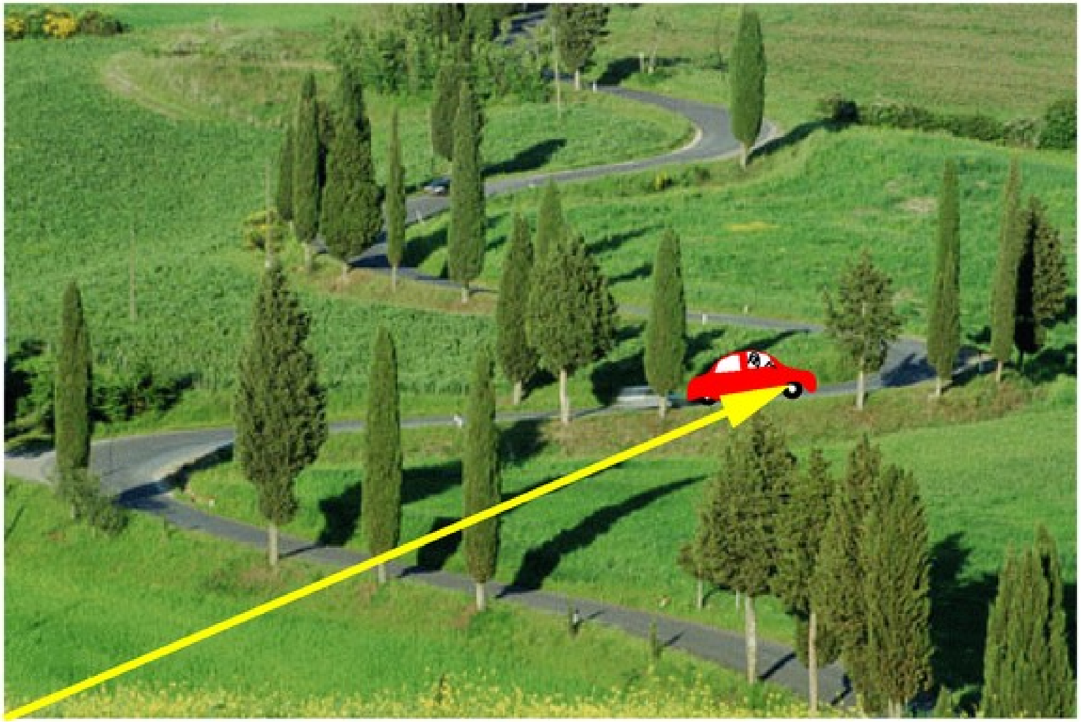
- Your experience of differentiation and integration has extended as far as *scalar* functions of single and multiple variables

$$\frac{d}{dx} f(x) \qquad \text{and} \qquad \frac{\partial}{\partial x} f(x, y, t)$$

- No surprise that we often wish to differentiate *vector functions*.
- For example, suppose you were driving along a wiggly road with position $\mathbf{r}(t)$ at time t.
- Differentiating $\mathbf{r}(t)$ should give velocity $\mathbf{v}(t)$.
- Differentiating $\mathbf{v}(t)$ should yield acceleration $\mathbf{a}(t)$.
- Differentiating $\mathbf{a}(t)$ should yield the jerk $\mathbf{j}(t)$.



Differentiation of a vector



Differentiation of a vector

- By analogy with the definition for a scalar function, the derivative of a vector function $\mathbf{a}(p)$ of a single parameter p is

$$\frac{d\mathbf{a}}{dp}(p) = \lim_{\delta p \to 0} \frac{\mathbf{a}(p+\delta p) - \mathbf{a}(p)}{\delta p} \ .$$

- If we write $\mathbf{a}$ in terms of components relative to a FIXED coordinate system ($\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}}$ constant)

$$\mathbf{a}(p) = a_1(p)\hat{\mathbf{\imath}} + a_2(p)\hat{\mathbf{\jmath}} + a_3(p)\hat{\mathbf{k}}$$

then

$$\frac{d\mathbf{a}}{dp}(p) = \frac{da_1}{dp}\hat{\mathbf{\imath}} + \frac{da_2}{dp}\hat{\mathbf{\jmath}} + \frac{da_3}{dp}\hat{\mathbf{k}} \ .$$

To differentiate a vector function defined wrt a fixed coordinate system, **differentiate each component separately**

All the familiar stuff works ...

- This means that
  - All the familiar rules of differentiation apply
  - they don't get munged by operations like scalar product and vector products.
- For example:

$$\frac{d}{dp}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dp} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dp}$$

$$\frac{d}{dp}(\mathbf{a} \cdot \mathbf{b}) = \frac{d\mathbf{a}}{dp} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\mathbf{b}}{dp} \quad .$$

- NB! (obvious really): $d\mathbf{a}/dp$ has
  - a different direction from **a**
  - a different magnitude from **a**.

Position, velocity and acceleration

- Suppose $\mathbf{r}(t)$ is the position vector of an object moving w.r.t. the orgin.

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$

- Then the instantaneous velocity is

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}$$

- and the acceleration is

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \quad .$$

Chain rule: more good news

- Likewise, the chain rule still applies.
- If $u = u(p)$:

$$\frac{d\mathbf{a}(p)}{dp} = \frac{d\mathbf{a}}{du}\frac{du}{dp}$$

- This follows directly from the fact that the vector derivative is just the vector of derivatives of the components.

♣ Example of chain rule

- The position of vehicle is given by $\mathbf{r}(u)$ where u is amount of fuel used by time t, so that $u = u(t)$.
- Its velocity must be

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{du}\frac{du}{dt}$$

- Its acceleration is

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{r}}{du^2}\left[\frac{du}{dt}\right]^2 + \frac{d\mathbf{r}}{du}\frac{d^2u}{dt^2}$$

♣ Example: direction of the derivative

Question

3D vector **a** has constant magnitude, but is varying over time.
What can you say about the direction of $d\mathbf{a}/dt$?

Answer

Using intuition: if only the direction is changing, then the vector must be tracing out points on the surface of a sphere. So $d\mathbf{a}/dt$ is orthogonal to **a**???

To prove this write

$$\frac{d}{dt}(\mathbf{a}\cdot\mathbf{a}) = \mathbf{a}\cdot\frac{d\mathbf{a}}{dt} + \frac{d\mathbf{a}}{dt}\cdot\mathbf{a} = 2\mathbf{a}\cdot\frac{d\mathbf{a}}{dt} \quad .$$

But $(\mathbf{a}\cdot\mathbf{a}) = a^2 = \text{const}$.

So

$$\frac{d}{dt}(\mathbf{a}\cdot\mathbf{a}) = 0 \qquad \Rightarrow 2\mathbf{a}\cdot\frac{d\mathbf{a}}{dt} = 0 \ \ \text{(QED)}$$

Integration of a vector function

- As with scalars, integration of a vector function of a single scalar variable is the reverse of differentiation.
- In other words

$$\int_{p_1}^{p_2} \left[\frac{d\mathbf{a}(p)}{dp} \right] dp = \mathbf{a}(p_2) - \mathbf{a}(p_1)$$

Eg, from dynamics-ville

$$\int_{t_1}^{t_2} \mathbf{a} \;\; dt = \mathbf{v}(t_2) - \mathbf{v}(t_1)$$

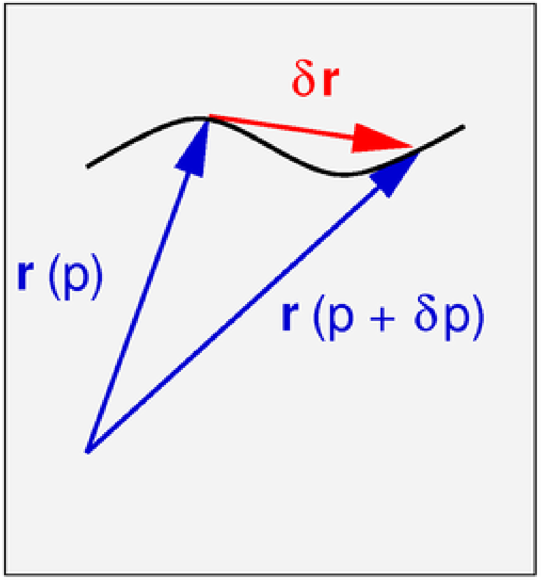
- However, other types of integral are possible, especially when the vector is a function of more than one variable.
- This requires the introduction of the concepts of scalar and vector fields.

See lecture 4!

Geometrical interpretation of derivatives

- Position vector $\mathbf{r}(p)$ traces a space curve.
- Vector $\delta\mathbf{r}$ is a secant to the curve
  $\delta\mathbf{r}/\delta p$ lies in the same direction as $\delta\mathbf{r}(p)$
- Take limit as $\delta p \to 0$

$d\mathbf{r}/dp$ is a tangent to the space curve



- Nothing special about the parameter p – may be various ways of parametrizing a particular curve.
- Consider helix aligned with z-axis. Could parametrize by for example:

  z, the "height" up the helix, or

  s, the "length" along the curve

Geometrical interpretation of derivatives /ctd

- If the parameter s is **arc-length** or **metric distance**, then we have:

$$|d\mathbf{r}| = ds$$

so

$$\left|\frac{d\mathbf{r}}{ds}\right| = 1$$

and

$d\mathbf{r}/ds$ is a **unit tangent** to $\mathbf{r}$ at s

- For s arc-length and p some other parametrization, we have

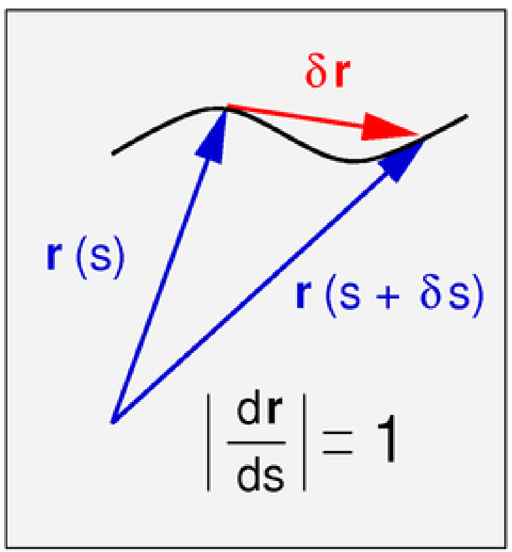
$$\frac{d\mathbf{r}}{dp} = \frac{d\mathbf{r}}{ds}\frac{ds}{dp}$$

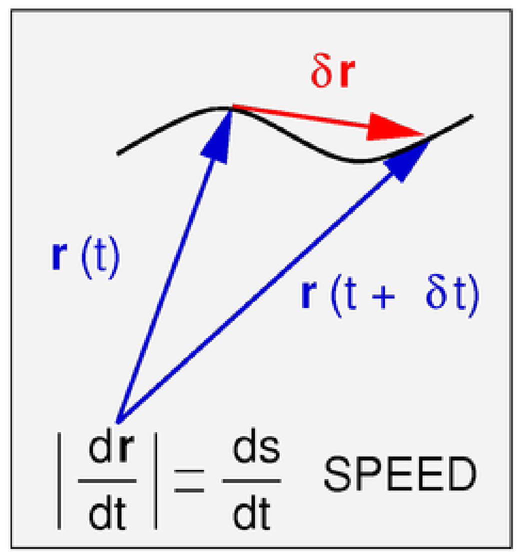
and

$$\left|\frac{d\mathbf{r}}{dp}\right| = \left|\frac{d\mathbf{r}}{ds}\right|\frac{ds}{dp} = \frac{ds}{dp}$$

Geometrical interpretation of derivatives /ctd

- To repeat, the derivative $d\mathbf{r}/dp$ is a vector
- Its direction is **always a tangent to curve** $\mathbf{r}(p)$
- Its magnitude is ds/dp, where s is arc length
- If the parameter is arc length s, then $d\mathbf{r}/ds$ is a **unit tangential vector**.
- If the parameter is time t, then magnitude $|d\mathbf{r}/dt|$ is the speed.





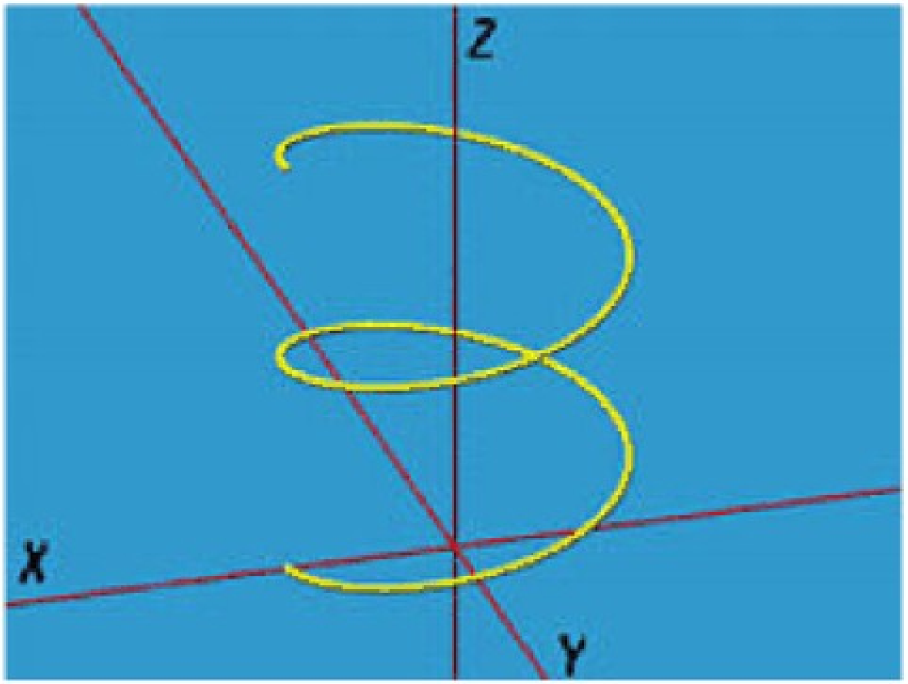
♣ Example

Question: Draw the curve

$$\mathbf{r} = a\cos\left(\frac{s}{\sqrt{a^2+h^2}}\right)\hat{\mathbf{i}} + a\sin\left(\frac{s}{\sqrt{a^2+h^2}}\right)\hat{\mathbf{j}} + \frac{hs}{\sqrt{a^2+h^2}}\hat{\mathbf{k}}$$

where s is arc length and h, a are constants.

Answer



♣ Example ctd

$$\mathbf{r} = a\cos\left(\frac{s}{\sqrt{a^2+h^2}}\right)\hat{\mathbf{i}} + a\sin\left(\frac{s}{\sqrt{a^2+h^2}}\right)\hat{\mathbf{j}} + \frac{hs}{\sqrt{a^2+h^2}}\hat{\mathbf{k}}$$

Show that the tangent $d\mathbf{r}/ds$ to the curve has a constant elevation angle w.r.t the xy-plane, and determine its magnitude.

Answer

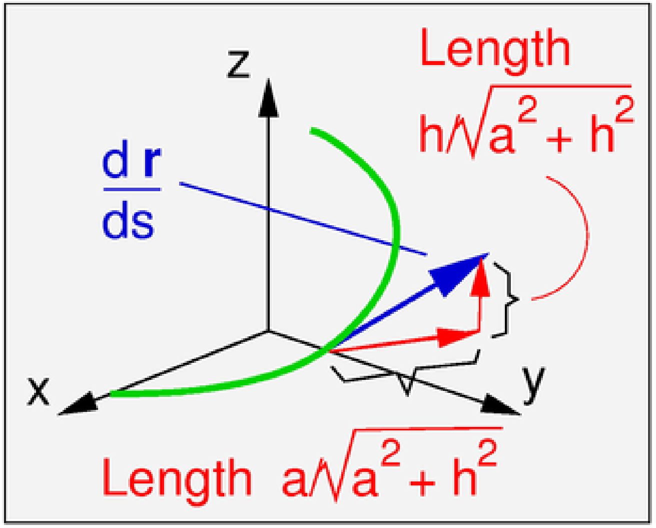
$$\frac{d\mathbf{r}}{ds} = -\frac{a}{\sqrt{a^2+h^2}}\sin()\hat{\mathbf{i}} + \frac{a}{\sqrt{a^2+h^2}}\cos()\hat{\mathbf{j}} + \frac{h}{\sqrt{a^2+h^2}}\hat{\mathbf{k}}$$

Projection on the xy plane has magnitude $a/\sqrt{a^2+h^2}$

Projection in the z direction $h/\sqrt{a^2+h^2}$

So the elevation angle is $\tan^{-1}(h/a)$, a constant.

We are expecting $|d\mathbf{r}/ds| = 1$, and indeed it is!



Why can't we say any old parameter is arc length?

- Arc length s parameter is special because $ds = |d\mathbf{r}|$,
- Or, in integral form, *whatever the parameter p*,

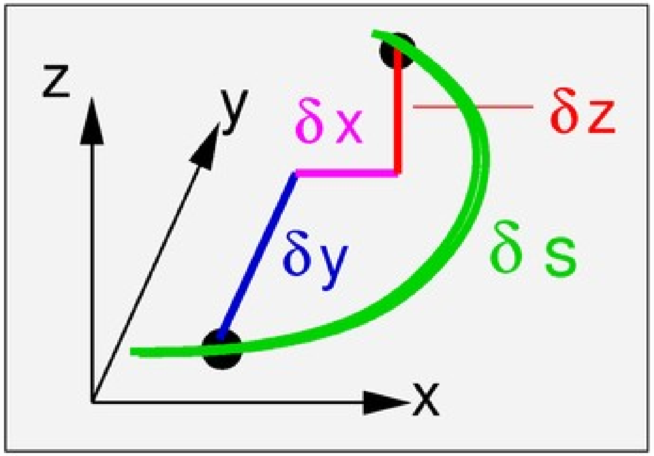
$$\text{Accumulated arc length} = \int_{p_0}^{p_1} \left| \frac{d\mathbf{r}}{dp} \right| dp \quad .$$

- Using Pythagoras' theorem on a short piece of curve. In the limit as ds tends to zero

$$ds^2 = dx^2 + dy^2 + dz^2 \; .$$

So if a curve is parameterized in terms of p

$$\frac{ds}{dp} = \sqrt{\left[\frac{dx}{dp}\right]^2 + \left[\frac{dy}{dp}\right]^2 + \left[\frac{dz}{dp}\right]^2} \; .$$



Arc length is special /ctd

- Suppose we had parameterized our helix as

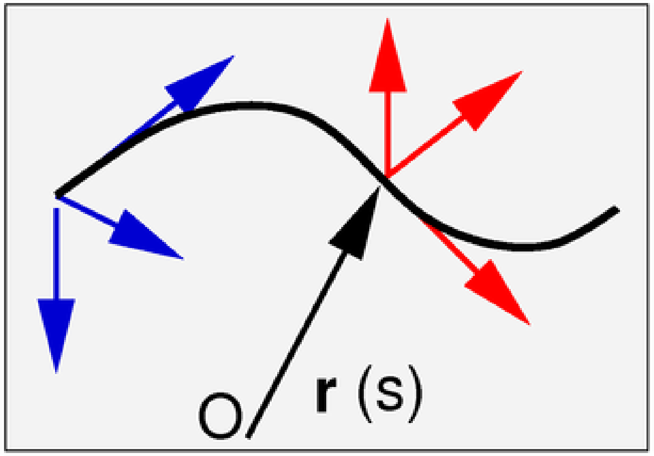
$$\mathbf{r} = a\cos p\,\hat{\mathbf{i}} \quad + \quad a\sin p\,\hat{\mathbf{j}} \quad + \quad hp\,\hat{\mathbf{k}}$$

- p is not arc length because

$$\left|\frac{d\mathbf{r}}{dp}\right| = \sqrt{\left[\frac{dx}{dp}\right]^2 + \left[\frac{dy}{dp}\right]^2 + \left[\frac{dz}{dp}\right]^2} \;=\; \sqrt{a^2\sin^2 p + a^2\cos^2 p + h^2}$$
$$= \sqrt{a^2 + h^2}$$
$$\neq 1$$

- So if we want to parameterize in terms of arclength, replace p with $s/\sqrt{a^2+h^2}$.

- Let's look more closely at parametrizing a 3D space curve in terms of arclength s.
- Introduce
  - orthogonal coord frames for each value s
  - each with its origin at $\mathbf{r}(s)$.
- To specify a coordinate frame we need
  - three mutually perpendicular directions
  - should be *intrinsic* to the curve
  - NOT fixed in an external reference frame.



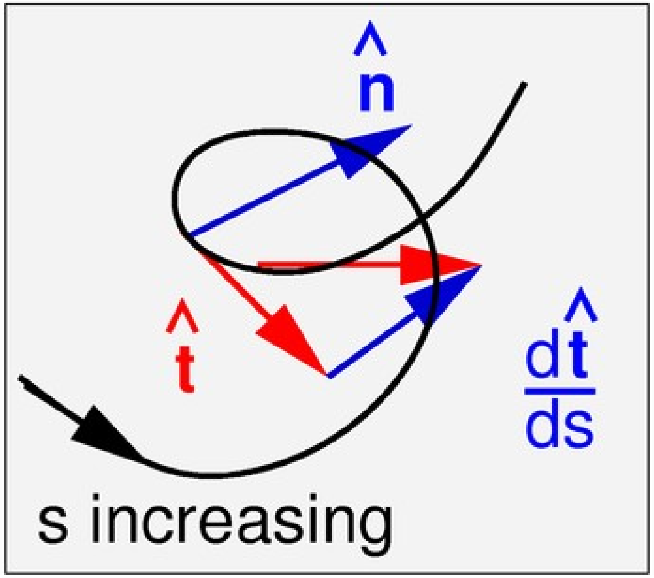
Curves in 3D

- Rollercoaster will help you see what's going on ...
- But it has a specially shaped rail or two rails that define the twists and turns.

- We are thinking about a 3D curve – just a 3D wire. Does the curve itself define its own twist and turns?

The Frénet-Serret Local Coordinates

Yes: method due to French mathematicians F-J. Frénet and J. A. Serret



1. Unit tangent $\hat{\mathbf{t}}$ Obvious choice is

$$\hat{\mathbf{t}} = d\mathbf{r}(s)/ds$$

2. Principal Normal $\hat{\mathbf{n}}$

Proved earlier that if $|\mathbf{a}(t)| = \text{const}$ then $\mathbf{a} \cdot d\mathbf{a}/dt = 0$. So

$$\hat{\mathbf{t}} = \hat{\mathbf{t}}(s), \quad |\hat{\mathbf{t}}| = \text{const} \Rightarrow \hat{\mathbf{t}} \cdot d\hat{\mathbf{t}}/ds = 0$$

Hence the principal normal $\hat{\mathbf{n}}$ is defined from

$$\kappa\hat{\mathbf{n}} = d\hat{\mathbf{t}}/ds$$

where $\kappa \geq 0$ is the curve's **curvature**.

3. The Binormal $\hat{\mathbf{b}}$

The third member of a local r-h set is the binormal, $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$.

Deriving the Frénet-Serret relationships

Tangent $\hat{\mathbf{t}}$, Normal $\hat{\mathbf{n}}$: $d\hat{\mathbf{t}}/ds = \kappa\hat{\mathbf{n}}$, Binormal $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$

- Since $\hat{\mathbf{b}} \cdot \hat{\mathbf{t}} = 0$, if we differentiate wrt s ...

$$\frac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}} + \hat{\mathbf{b}} \cdot \frac{d\hat{\mathbf{t}}}{ds} \;=\; \frac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}} + \hat{\mathbf{b}} \cdot \kappa\hat{\mathbf{n}} = 0$$

from which

$$\frac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}} = 0.$$

- This means that $d\hat{\mathbf{b}}/ds$ is along the direction of $\hat{\mathbf{n}}$:

$$\frac{d\hat{\mathbf{b}}}{ds} = -\tau(s)\hat{\mathbf{n}}(s)$$

where τ is the space curve's **torsion**.

Deriving the Frénet-Serret relationships

Tangent $\hat{\mathbf{t}}$, Normal $\hat{\mathbf{n}}$, Binormal $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$
$d\hat{\mathbf{t}}/ds = \kappa\hat{\mathbf{n}}$, $d\hat{\mathbf{b}}/ds = -\tau(s)\hat{\mathbf{n}}(s)$

- Differentiating $\hat{\mathbf{n}} \cdot \hat{\mathbf{t}} = 0$:

$$\begin{aligned}
(d\hat{\mathbf{n}}/ds) \cdot \hat{\mathbf{t}} + \hat{\mathbf{n}} \cdot (d\hat{\mathbf{t}}/ds) &= 0 \\
(d\hat{\mathbf{n}}/ds) \cdot \hat{\mathbf{t}} + \hat{\mathbf{n}} \cdot \kappa\hat{\mathbf{n}} &= 0 \\
(d\hat{\mathbf{n}}/ds) \cdot \hat{\mathbf{t}} &= -\kappa
\end{aligned}$$

- Now do the same to $\hat{\mathbf{n}} \cdot \hat{\mathbf{b}} = 0$:

$$\begin{aligned}
(d\hat{\mathbf{n}}/ds) \cdot \hat{\mathbf{b}} + \hat{\mathbf{n}} \cdot (d\hat{\mathbf{b}}/ds) &= 0 \\
(d\hat{\mathbf{n}}/ds) \cdot \hat{\mathbf{b}} + \hat{\mathbf{n}} \cdot (-\tau)\hat{\mathbf{n}} &= 0 \\
(d\hat{\mathbf{n}}/ds) \cdot \hat{\mathbf{b}} &= +\tau
\end{aligned}$$

- HENCE

$$\frac{d\hat{\mathbf{n}}}{ds} = -\kappa(s)\hat{\mathbf{t}}(s) + \tau(s)\hat{\mathbf{b}}(s).$$

Summary of the Frénet-Serret relationships

These three expressions are called the Frénet-Serret relationships:

- $d\hat{\mathbf{t}}/ds = \kappa\hat{\mathbf{n}}$
- $d\hat{\mathbf{n}}/ds = -\kappa(s)\hat{\mathbf{t}}(s) + \tau(s)\hat{\mathbf{b}}(s)$
- $d\hat{\mathbf{b}}/ds = -\tau(s)\hat{\mathbf{n}}(s)$

- They describe by how much the intrinsic coordinate system changes orientation as we move along a space curve.

Question Derive $\kappa(s)$ and $\tau(s)$ for the curve

$$\mathbf{r}(s) = a\cos(s/\beta)\,\hat{\mathbf{\imath}} + a\sin(s/\beta)\,\hat{\mathbf{\jmath}} + h\,(s/\beta)\,\hat{\mathbf{k}}$$

where $\beta = \sqrt{a^2 + h^2}$

Answer:

- Denote sin, $\cos(s/\beta)$ as $\mathcal{S}$ and $\mathcal{C}$.

  We found the unit tangent earlier as

$$\hat{\mathbf{t}} = (d\mathbf{r}/ds) = [-(a/\beta)\,\mathcal{S}, \quad (a/\beta)\,\mathcal{C}, \quad (h/\beta)] \ .$$

- Hence

$$\kappa\hat{\mathbf{n}} = \left(d\hat{\mathbf{t}}/ds\right) = \left[-\left(a/\beta^2\right)\mathcal{C}, \quad -\left(a/\beta^2\right)\mathcal{S}, \quad 0\right]$$

- The curvature must be positive, so

$$\kappa = \left(a/\beta^2\right) \qquad \hat{\mathbf{n}} = [-\mathcal{C}, \quad -\mathcal{S}, \quad 0] \ .$$

- So the curvature is constant, and $\hat{\mathbf{n}}$ parallel to the xy-plane.

♣ Example /continued

- Recall

$$\hat{\mathbf{t}} = [-(a/\beta)\,\mathcal{S}, \quad (a/\beta)\,\mathcal{C}, \quad (h/\beta)] \qquad\qquad \hat{\mathbf{n}} = [-\mathcal{C}, \quad -\mathcal{S}, \quad 0] \ .$$

- So the binormal is

$$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ (-a/\beta)\mathcal{S} & (a/\beta)\mathcal{C} & (h/\beta) \\ -\mathcal{C} & -\mathcal{S} & 0 \end{vmatrix} = \left[\left(\frac{h}{\beta}\right)\mathcal{S}, \ -\left(\frac{h}{\beta}\right)\mathcal{C}, \ \left(\frac{a}{\beta}\right) \right]$$

- Hence

$$(d\hat{\mathbf{b}}/ds) = \left[(h/\beta^2)\,\mathcal{C}, \quad (h/\beta^2)\,\mathcal{S}, \quad 0 \right] = (-h/\beta^2)\,\hat{\mathbf{n}}$$

- So the torsion

$$\tau = (h/\beta^2)$$

again a constant.

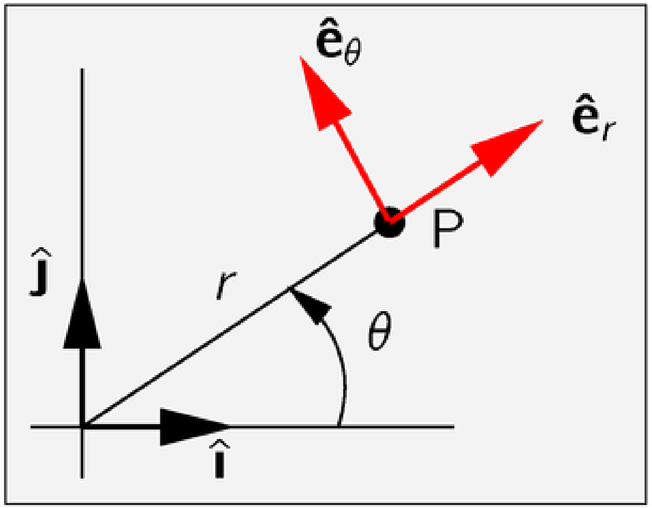
Derivative (eg velocity) components in plane polars

In plane polar coordinates, the radius vector of any point P is given by

$$\mathbf{r} = r(\cos\theta\hat{\mathbf{\imath}} + \sin\theta\hat{\mathbf{\jmath}}) = r\hat{\mathbf{e}}_r$$

where we have introduced the unit radial vector

$$\hat{\mathbf{e}}_r = \cos\theta\hat{\mathbf{\imath}} + \sin\theta\hat{\mathbf{\jmath}} \quad .$$



The other "natural" unit vector in plane polars is orthogonal to $\hat{\mathbf{e}}_r$ and is

$$\hat{\mathbf{e}}_\theta = -\sin\theta\hat{\mathbf{\imath}} + \cos\theta\hat{\mathbf{\jmath}}$$

so that $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_r = \hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_\theta = 1$ and $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_\theta = 0$.

Aside: notation

- Some texts will use the notation

$$\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}$$

  to denote unit vectors in the radial and tangential directions

- I prefer the more general notation

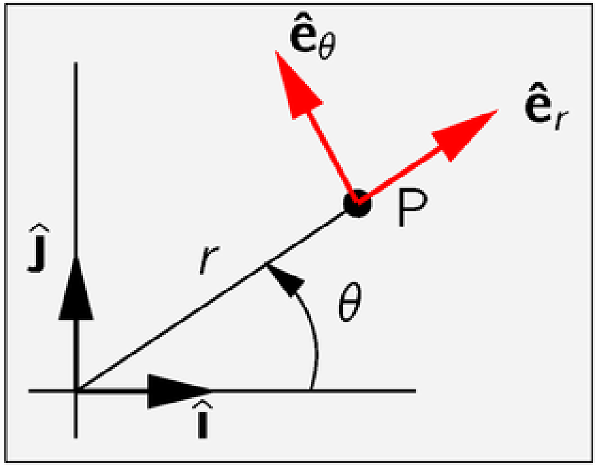
$$\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta$$

  (as used in, eg, Riley).

- You should be familiar and comfortable with either

Derivative (eg velocity) components in plane polars

- Now suppose P is moving so that $\mathbf{r}$ is a function of time t.



- Its velocity is

$$
\begin{aligned}
\dot{\mathbf{r}} = \frac{d}{dt}(r\hat{\mathbf{e}}_r) &= \frac{dr}{dt}\hat{\mathbf{e}}_r + r\frac{d\hat{\mathbf{e}}_r}{dt} \\
&= \frac{dr}{dt}\hat{\mathbf{e}}_r + r\frac{d\theta}{dt}(-\sin\theta\hat{\mathbf{\imath}} + \cos\theta\hat{\mathbf{\jmath}}) \\
&= \frac{dr}{dt}\hat{\mathbf{e}}_r + r\frac{d\theta}{dt}\hat{\mathbf{e}}_\theta \\
&= \textbf{radial} + \textbf{tangential}
\end{aligned}
$$

- Note that

$$
\frac{d\hat{\mathbf{e}}_r}{dt} = \frac{d\theta}{dt}\hat{\mathbf{e}}_\theta \qquad \frac{d\hat{\mathbf{e}}_\theta}{dt} = \frac{d}{dt}(-\sin\theta\hat{\mathbf{\imath}} + \cos\theta\hat{\mathbf{\jmath}}) = -\frac{d\theta}{dt}\hat{\mathbf{e}}_r
$$

Acceleration components in plane polars

- Recap ...

$$\dot{\mathbf{r}} = \frac{dr}{dt}\hat{\mathbf{e}}_r + r\frac{d\theta}{dt}\hat{\mathbf{e}}_\theta \; ; \qquad \frac{d\hat{\mathbf{e}}_r}{dt} = \frac{d\theta}{dt}\hat{\mathbf{e}}_\theta \; ; \qquad \frac{d\hat{\mathbf{e}}_\theta}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{e}}_r$$

- Differentiating $\dot{\mathbf{r}}$ gives the accel. of P

$$\ddot{\mathbf{r}} = \frac{d^2r}{dt^2}\hat{\mathbf{e}}_r + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\mathbf{e}}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\mathbf{e}}_\theta + r\frac{d^2\theta}{dt^2}\hat{\mathbf{e}}_\theta - r\frac{d\theta}{dt}\frac{d\theta}{dt}\hat{\mathbf{e}}_r$$

$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{e}}_r \; + \; \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\hat{\mathbf{e}}_\theta$$

Acceleration components in plane polars

- We just saw

$$\ddot{\mathbf{r}} = \left[\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{e}}_r \quad + \quad \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\hat{\mathbf{e}}_\theta$$

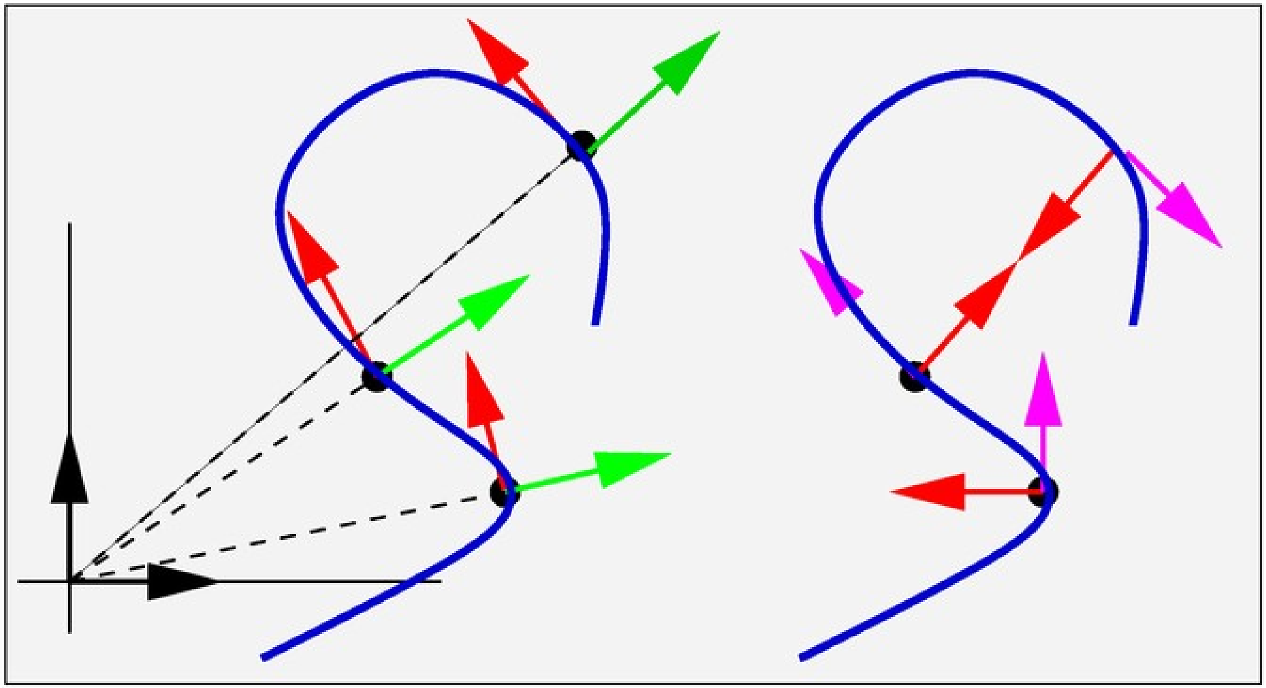
- Three obvious cases:

$$\theta \text{ const}: \quad \ddot{\mathbf{r}} = \frac{d^2 r}{dt^2}\hat{\mathbf{e}}_r$$

$$r \text{ const}: \quad \ddot{\mathbf{r}} = -r\left(\frac{d\theta}{dt}\right)^2\hat{\mathbf{e}}_r + r\frac{d^2\theta}{dt^2}\hat{\mathbf{e}}_\theta$$

$$r \text{ and } d\theta/dt \text{ const}: \ddot{\mathbf{r}} = -r\left(\frac{d\theta}{dt}\right)^2\hat{\mathbf{e}}_r$$

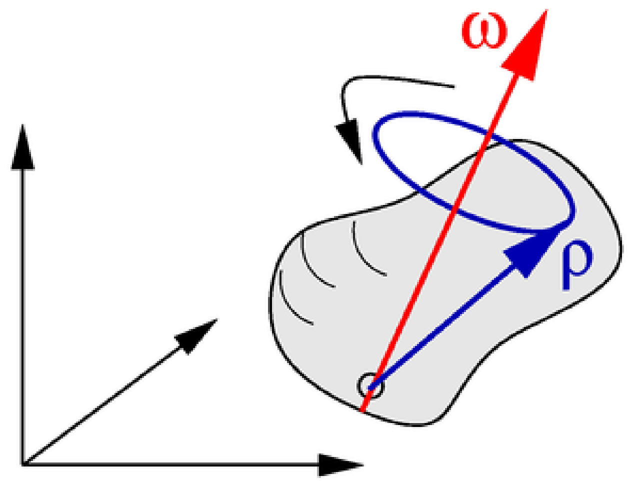
Fixed, varying, and instrinsic coordinates



Rotating systems

- Body rotates with constant $\boldsymbol{\omega}$ about axis passing through the body origin.

  Assume the body origin is fixed. We observe from a fixed coord system $Oxyz$.



- If $\boldsymbol{\rho}$ is a vector of constant mag and constant direction in the rotating system, then in the fixed system it must be a function of t.

$$\mathbf{r}(t) = \mathrm{R}(t)\boldsymbol{\rho} \qquad \Rightarrow \qquad \frac{d\mathbf{r}}{dt} = \dot{\mathrm{R}}\boldsymbol{\rho} = \dot{\mathrm{R}}\mathrm{R}^{\top}\mathbf{r}$$

  * $d\mathbf{r}/dt$ will have fixed magnitude;
  * $d\mathbf{r}/dt$ will always be perpendicular to the axis of rotation;
  * $d\mathbf{r}/dt$ will vary in direction within those constraints;
  * $\mathbf{r}(t)$ will move in a plane in the fixed system.

Consider the term $\dot{\mathtt{R}}\mathtt{R}^\top$

- Note that $\mathtt{R}\mathtt{R}^\top = \mathtt{I}$, hence

$$\begin{aligned} \dot{\mathtt{R}}\mathtt{R}^\top + \mathtt{R}\dot{\mathtt{R}}^\top &= 0 \\ \dot{\mathtt{R}}\mathtt{R}^\top &= -\mathtt{R}\dot{\mathtt{R}}^\top \end{aligned}$$

- Thus $\dot{\mathtt{R}}\mathtt{R}^\top$ is anti-symmetric:

$$\dot{\mathtt{R}}\mathtt{R}^\top = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

- Application of a matrix of this form to an arbitrary vector has **precisely the same effect** as the cross product operator, $\boldsymbol{\omega}\times$, where $\boldsymbol{\omega} = [xyz]^\top$.
- Thus

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$$

Rotating co-ordinate systems 2

- Now $\boldsymbol{\rho}$ is the position vector of a point P in the rotating body, but which is moving too, with respect to the rotating system

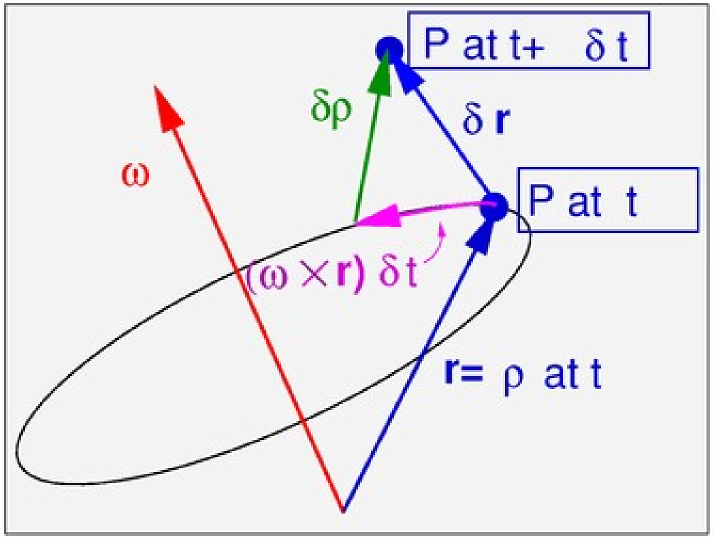
$$\mathbf{r}(t) = \mathrm{R}(t)\boldsymbol{\rho}(t)$$

- Differentiating with respect to time:

$$\frac{d\mathbf{r}}{dt} = \dot{\mathrm{R}}\boldsymbol{\rho} + \mathrm{R}\dot{\boldsymbol{\rho}} = \dot{\mathrm{R}}\mathrm{R}^{\top}\mathbf{r} + \mathrm{R}\dot{\boldsymbol{\rho}}$$

- The **instantaneous velocity** of P in the fixed frame is

$$\frac{d\mathbf{r}}{dt} = \mathrm{R}\dot{\boldsymbol{\rho}} + \boldsymbol{\omega} \times \mathbf{r}$$



- Second term is contribution from the rotating frame
- First term is linear velocity in the rotating frame, referred to the fixed frame

- Now consider **second** differential:

$$\ddot{\mathbf{r}} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}} + \dot{\mathrm{R}}\dot{\boldsymbol{\rho}} + \mathrm{R}\ddot{\boldsymbol{\rho}}$$

- If angular velocity constant, first term is zero
- Now substituting for $\dot{\mathbf{r}}$ we have

$$\begin{aligned}\ddot{\mathbf{r}} &= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r} + \mathrm{R}\dot{\boldsymbol{\rho}}) + \dot{\mathrm{R}}\dot{\boldsymbol{\rho}} + \mathrm{R}\ddot{\boldsymbol{\rho}} \\ &= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times \mathrm{R}\dot{\boldsymbol{\rho}} + \dot{\mathrm{R}}\mathrm{R}^\top\mathrm{R}\dot{\boldsymbol{\rho}} + \mathrm{R}\ddot{\boldsymbol{\rho}} \\ &= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times \mathrm{R}\dot{\boldsymbol{\rho}} + \boldsymbol{\omega} \times \mathrm{R}\dot{\boldsymbol{\rho}} + \mathrm{R}\ddot{\boldsymbol{\rho}} \\ &= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times (\mathrm{R}\dot{\boldsymbol{\rho}}) + \mathrm{R}\ddot{\boldsymbol{\rho}}\end{aligned}$$

- The **instantaneous acceleration** is therefore

$$\ddot{\mathbf{r}} = \mathrm{R}\ddot{\boldsymbol{\rho}} + 2\boldsymbol{\omega} \times (\mathrm{R}\dot{\boldsymbol{\rho}}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

- The **instantaneous acceleration** is

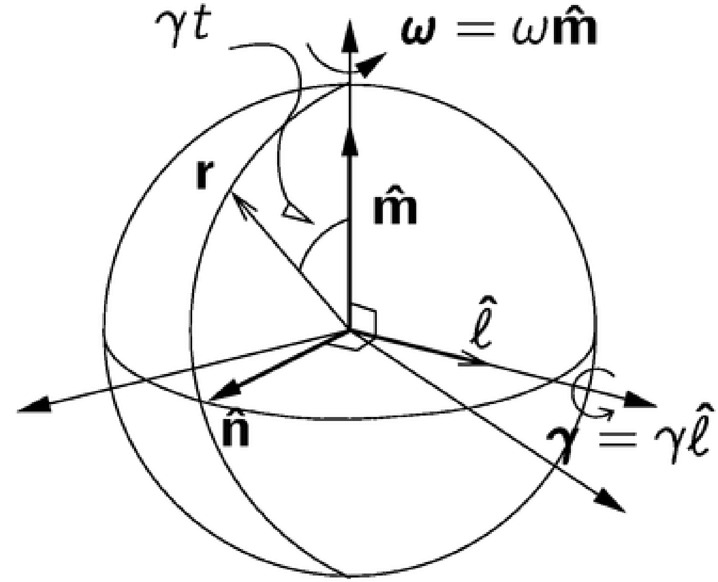
$$\ddot{\mathbf{r}} = \mathrm{R}\ddot{\boldsymbol{\rho}} + 2\boldsymbol{\omega} \times (\mathrm{R}\dot{\boldsymbol{\rho}}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

* Term 1 is P's acceleration in the rotating frame.
* Term 3 is the centripetal accel: magnitude $\omega^2 r$ and direction $-\mathbf{r}$.
* Term 2 is a SURPRISE!

  It is a coupling of rotation and velocity of P in the rotating frame.
  It is the **Coriolis acceleration**.

Q Find the instantaneous acceleration as observed in a fixed frame of a projectile fired along a line of longitude (with angular velocity of $\boldsymbol{\gamma}$ constant relative to the sphere) if the sphere is rotating with angular velocity $\boldsymbol{\omega}$.
A In the rotating frame

$$\begin{aligned}\dot{\boldsymbol{\rho}} &= \boldsymbol{\gamma} \times \boldsymbol{\rho} \\ \ddot{\boldsymbol{\rho}} &= \boldsymbol{\gamma} \times \dot{\boldsymbol{\rho}} \\ &= \boldsymbol{\gamma} \times (\boldsymbol{\gamma} \times \boldsymbol{\rho})\end{aligned}$$



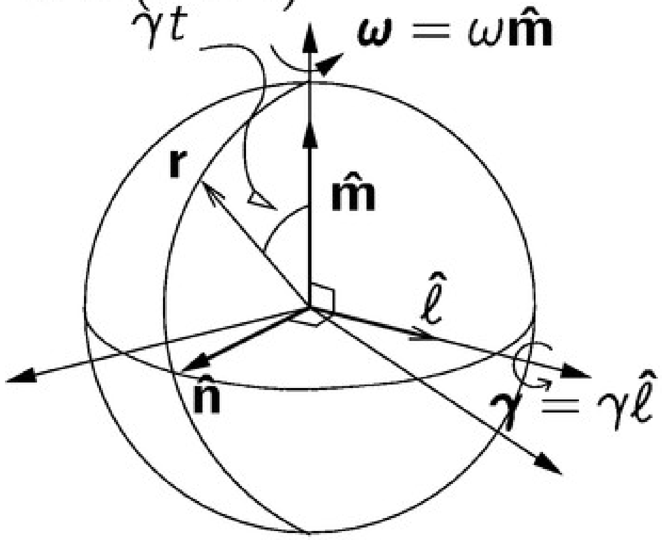
In fixed frame, instantaneous acceleration:

$$\ddot{\mathbf{r}} = \boldsymbol{\gamma} \times (\boldsymbol{\gamma} \times \mathbf{r}) + 2\boldsymbol{\omega} \times (\boldsymbol{\gamma} \times \mathbf{r}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

In rotating frm + Coriolis + Centripetal

♣ Example /ctd

Repeated: $\ddot{\mathbf{r}} = \boldsymbol{\gamma} \times (\boldsymbol{\gamma} \times \mathbf{r}) + 2\boldsymbol{\omega} \times (\boldsymbol{\gamma} \times \mathbf{r}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$



1) As $\boldsymbol{\gamma} = \gamma\hat{\ell}$, $\boldsymbol{\rho} = R\cos(\gamma t)\hat{\mathbf{m}} + R\sin(\gamma t)\hat{\mathbf{n}}$
⇒acceleration in rotating frame is

$$\boldsymbol{\gamma} \times (\boldsymbol{\gamma} \times \boldsymbol{\rho}) = -\gamma^2\mathbf{r}$$

2) Centripetal accel due to rotation of sphere is

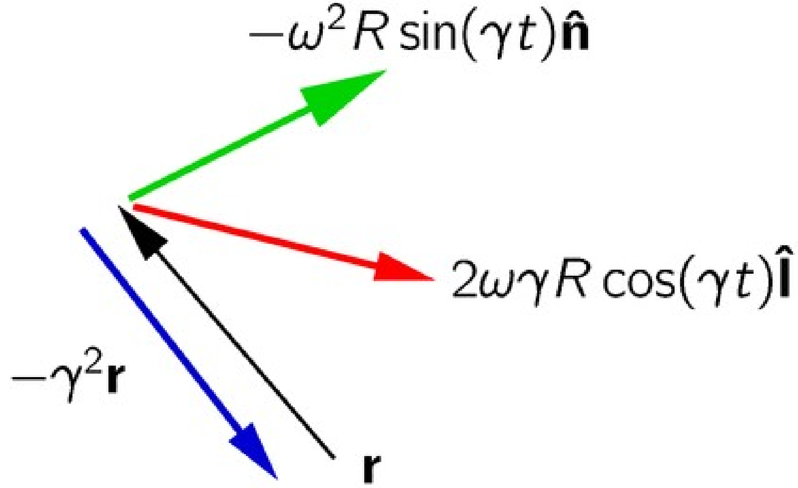
$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = -\omega^2 R\sin(\gamma t)\hat{\mathbf{n}}$$

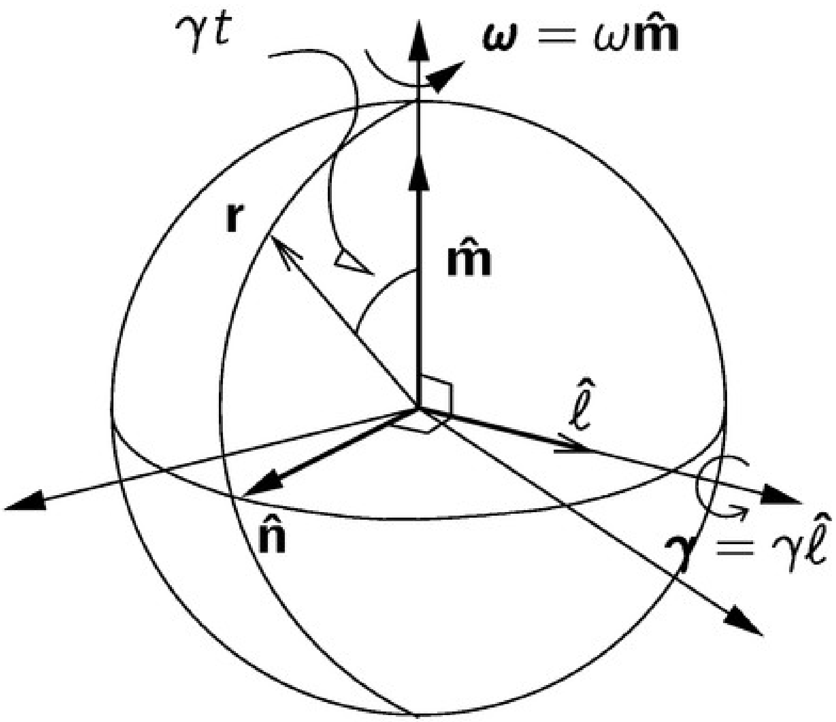
3) The Coriolis acceleration is

$$2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}} = 2\begin{bmatrix}0\\ \omega\\ 0\end{bmatrix} \times \left(\begin{bmatrix}\gamma\\ 0\\ 0\end{bmatrix} \times \begin{bmatrix}0\\ R\cos(\gamma t)\\ R\sin(\gamma t)\end{bmatrix}\right) = 2\omega\gamma R\cos(\gamma t)\hat{\ell}$$

Recap:

- Accel in rotating frame $-\boldsymbol{\gamma}^2\mathbf{r}$
- Centripetal due to sphere rotating $-\omega^2 R\sin(\gamma t)\hat{\mathbf{n}}$
- Coriolis acceleration: $2\omega\gamma R\cos(\gamma t)\hat{\mathbf{l}}$



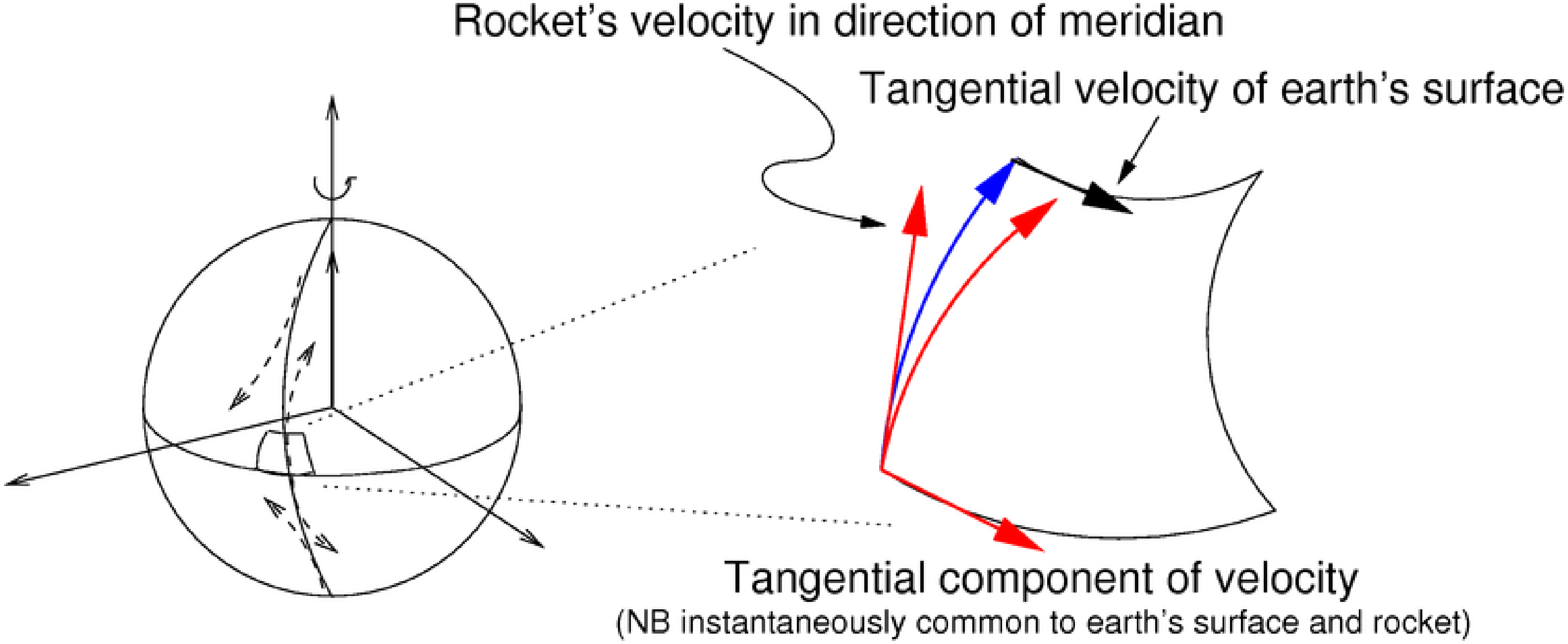


♣ Example /ctd

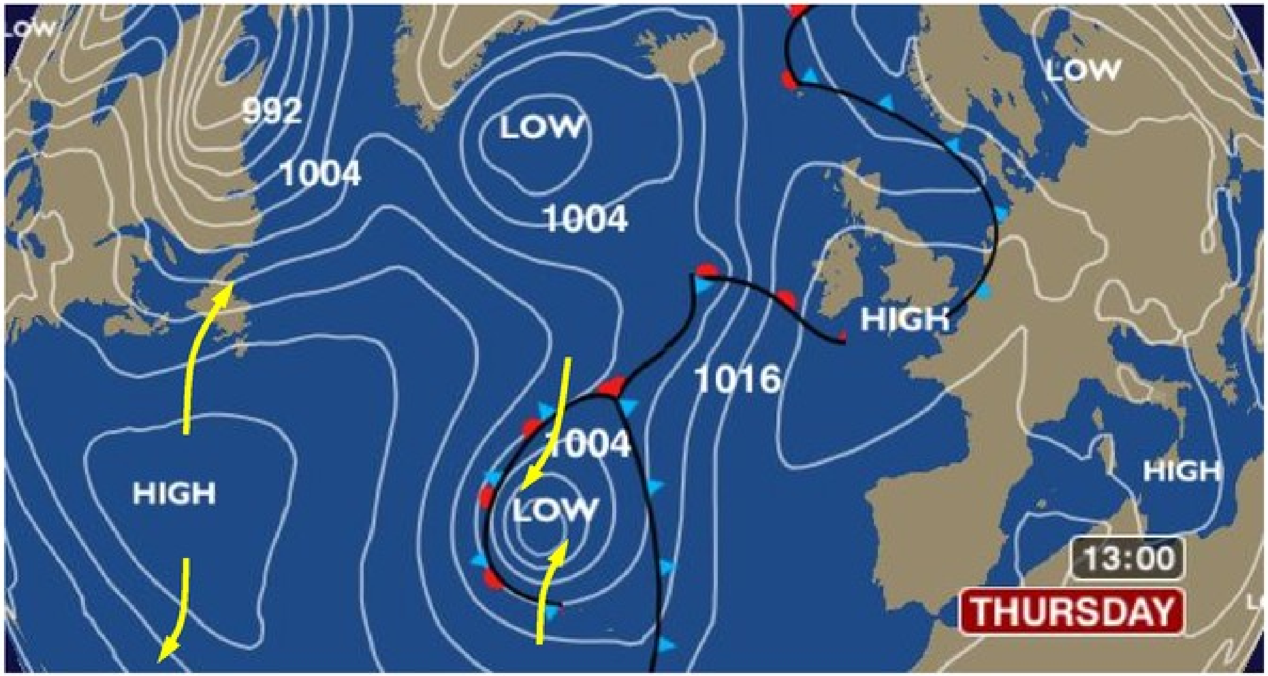
- Consider a rocket on rails which stretch north from the equator.
- As rocket travels **north** it experiences the Coriolis force exerted by the rails:

2	γ	ω	$R\cos(\gamma t)$	$\hat{\ell}$
+ve	-ve	+ve	+ve	

- Coriolis force is in the direction opposed to $\hat{\ell}$ (i.e. opposing earth's rotation).



- Because of the rotation of the earth, the Coriolis acceleration is of great importance in meteorology



♣ Coriolis acceleration

Summary

- We started by differentiating vectors wrt to a fixed coordinate system.
- Then looked at the properties of the derivative of a position vector **r** with respect to a general parameter p and the special parameters of arc-length s, and time t
- considered derivatives with respect to other coordinate systems, in particular those of the position vector in polar coordinates with respect to time.
- derived Frénet-Serret relationships — a method of describing a 3D space curve by describing the change in a intrinsic coordinate system as it moves along the curve.
- discussed rotating coordinate systems; we saw that there is coupled term in the acceleration, called the Coriolis acceleration.