2. More Algebra & Geometry using Vectors

In which we discuss ...

- Vector products:

  Scalar Triple Product, Vector Triple Product, Vector Quadruple Product
- Geometry of Lines and Planes
- Solving vector equations
- Angular velocity and moments

Triple and multiple products

- Using mixtures of scalar products and vector products, it is possible to derive
  - "triple products" between three vectors
  - *n*-products between *n* vectors.
- Nothing new about these
  — but some have nice geometric interpretations ...
- We will look at the
  - Scalar triple product
  - Vector triple product
  - Vector quadruple product

Scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

- Scalar triple product given by the true determinant

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- Your knowledge of determinants tells you that if you
  - swap one pair of rows of a determinant, sign changes;
  - swap two pairs of rows, its sign stays the same.

- Hence

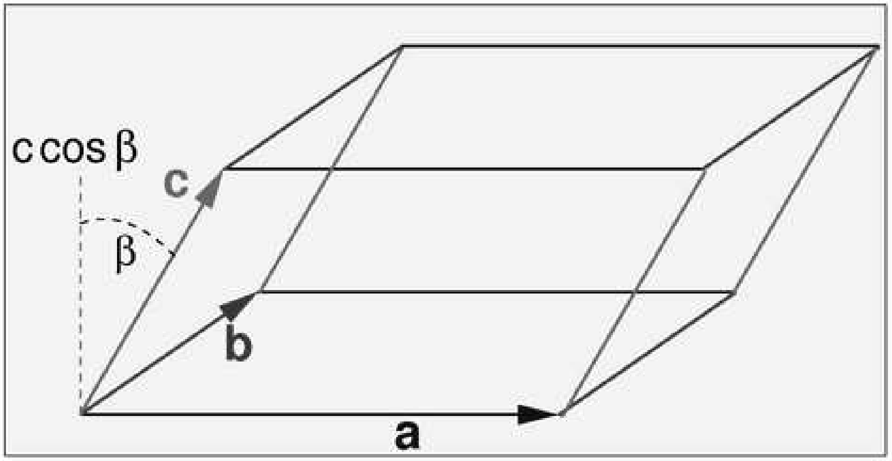
(i) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$ (Cyclic permutation.)

(ii) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$ and so on. (Anti-cyclic permutation)

(iii) The fact that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ allows the scalar triple product to be written as $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$.
This notation is not very helpful, and we will try to avoid it below.

Geometrical interpretation of scalar triple product

- The scalar triple product gives the volume of the parallelopiped whose sides are represented by the vectors **a**, **b**, and **c**.



- Vector product $(\mathbf{a} \times \mathbf{b})$ has

  magnitude equal to the area of the base

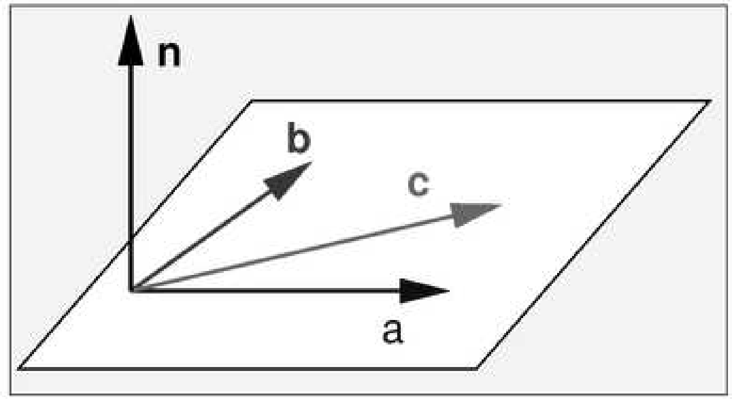
  direction perpendicular to the base.

- The *component* of **c** in this direction is equal to the height of the parallelopiped

  Hence

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = \text{volume of parallelopied}$$

Linearly dependent vectors



- If the scalar triple product of three vectors

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$$

then the vectors are **linearly dependent**.

$$\mathbf{a} = \lambda \mathbf{b} + \mu \mathbf{c}$$

- You can see this immediately either using the determinant

— The determinant would have one row that was a linear combination of the others

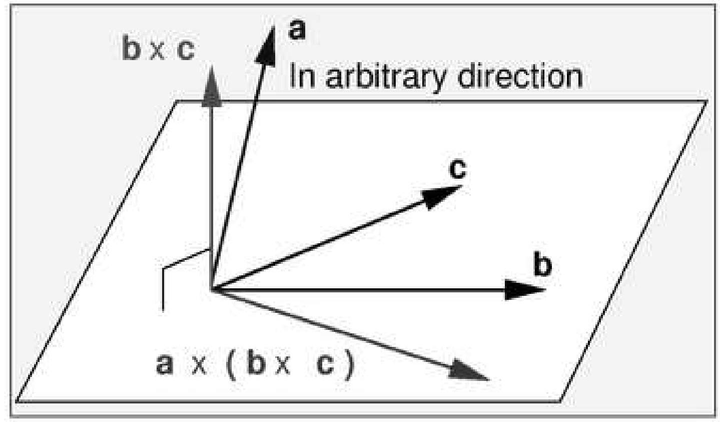
- or geometrically for a 3-dimensional vector.

— the parallelopiped would have zero volume if squashed flat.

Vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is perpendicular to $(\mathbf{b} \times \mathbf{c})$ but $(\mathbf{b} \times \mathbf{c})$ is perpendicular to $\mathbf{b}$ *and* $\mathbf{c}$. So $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ must be *coplanar* with $\mathbf{b}$ and $\mathbf{c}$.

$$\Rightarrow \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \lambda \mathbf{b} + \mu \mathbf{c}$$

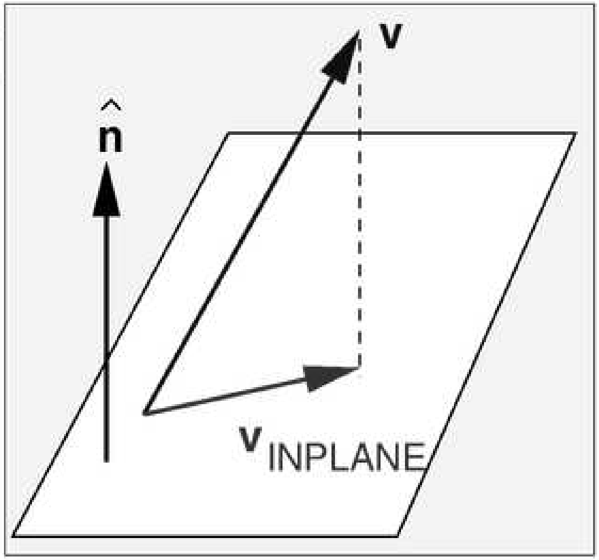


$$\begin{aligned}
(\mathbf{a} \times (\mathbf{b} \times \mathbf{c}))_1 &= a_2(\mathbf{b} \times \mathbf{c})_3 - a_3(\mathbf{b} \times \mathbf{c})_2 \\
&= a_2(b_1c_2 - b_2c_1) + a_3(b_1c_3 - b_3c_1) \\
&= (a_2c_2 + a_3c_3)b_1 - (a_2b_2 + a_3b_3)c_1 \\
&= (a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1 \\
&= (\mathbf{a} \cdot \mathbf{c})b_1 - (\mathbf{a} \cdot \mathbf{b})c_1
\end{aligned}$$

Similarly for components 2 and 3: so

$$[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})] = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

Projection using vector triple product



- Books say that the vector projection of any old vector $\mathbf{v}$ into a plane with normal $\hat{\mathbf{n}}$ is
$$\mathbf{v}_{\mathrm{INPLANE}} = \hat{\mathbf{n}} \times (\mathbf{v} \times \hat{\mathbf{n}}).$$
- The component of $\mathbf{v}$ in the $\hat{\mathbf{n}}$ direction is $\mathbf{v} \cdot \hat{\mathbf{n}}$ so I would write the vector projection as
$$\mathbf{v}_{\mathrm{INPLANE}} = \mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$$
- Can we reconcile the two expressions? Subst. $\hat{\mathbf{n}} \leftarrow \mathbf{a}$, $\mathbf{v} \leftarrow \mathbf{b}$, $\hat{\mathbf{n}} \leftarrow \mathbf{c}$, into our earlier formula
$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\ \hat{\mathbf{n}} \times (\mathbf{v} \times \hat{\mathbf{n}}) &= (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\mathbf{v} - (\hat{\mathbf{n}} \cdot \mathbf{v})\hat{\mathbf{n}} \\ &= \mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} \end{aligned}$$
- Fantastico! But $\mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$ is much easier to understand, cheaper to compute!

Vector Quadruple Product $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$

- We have just learned that

$$\begin{aligned} [\mathbf{p} \times (\mathbf{q} \times \mathbf{r})] &= (\mathbf{p} \cdot \mathbf{r})\mathbf{q} - (\mathbf{p} \cdot \mathbf{q})\mathbf{r} \\ \Rightarrow \quad (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= ?? \end{aligned}$$

- Regarding $\mathbf{a} \times \mathbf{b}$ as a single vector
  $\Rightarrow$ vqp must be a linear combination of $\mathbf{c}$ and $\mathbf{d}$
- Regarding $\mathbf{c} \times \mathbf{d}$ as a single vector
  $\Rightarrow$ vqp must be a linear combination of $\mathbf{a}$ and $\mathbf{b}$.
- Substituting in carefully (you check ...)

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\mathbf{c} - [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]\mathbf{d} \\ &= [(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{a}]\mathbf{b} - [(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{b}]\mathbf{a} \end{aligned}$$

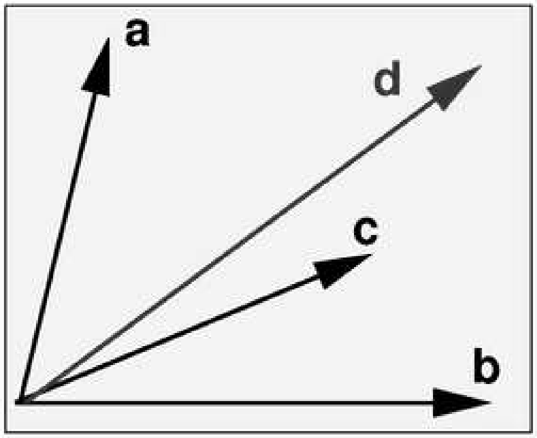
- Using just the R-H sides of what we just wrote ...

$$[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]\,\mathbf{d} = [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d}]\,\mathbf{a} + [(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{d}]\,\mathbf{b} + [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\,\mathbf{c}$$

- So

$$\begin{aligned} \mathbf{d} &= \frac{[(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d}]\,\mathbf{a} + [(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{d}]\,\mathbf{b} + [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\,\mathbf{c}}{[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]} \\ &= \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} \quad . \end{aligned}$$

- Don't remember by ♡
- **Key point is that the projection of a 3D vector d onto a basis set of 3 non-coplanar vectors is UNIQUE.**



♣ Example

Question

Use the quadruple vector product to express the vector $\mathbf{d} = [3, 2, 1]$ in terms of the vectors $\mathbf{a} = [1, 2, 3]$, $\mathbf{b} = [2, 3, 1]$ and $\mathbf{c} = [3, 1, 2]$.

Answer

$$\mathbf{d} = \frac{[(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d}]\,\mathbf{a} + [(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{d}]\,\mathbf{b} + [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\,\mathbf{c}}{[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]}$$

So, grinding away at the determinants, we find

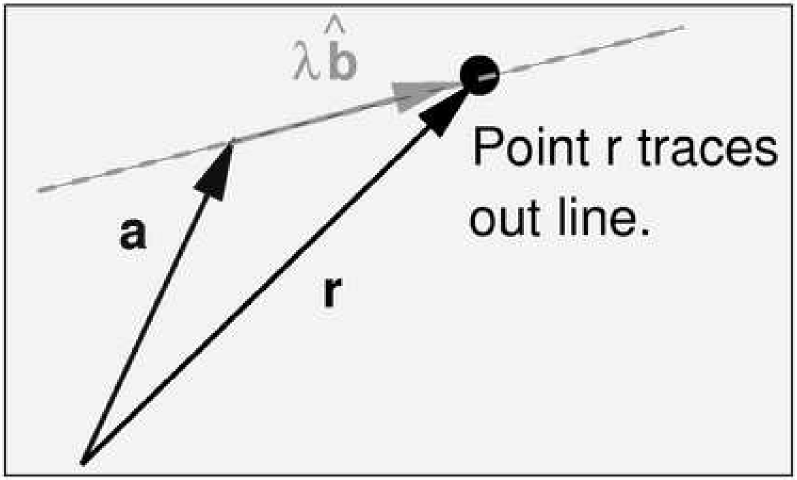
- $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -18$ and $(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d} = 6$
- $(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{d} = -12$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d} = -12$.

So

$$\begin{aligned} \mathbf{d} &= \frac{1}{-18}(6\mathbf{a} - 12\mathbf{b} - 12\mathbf{c}) \\ &= \frac{1}{3}(-\mathbf{a} + 2\mathbf{b} + 2\mathbf{c}) \end{aligned}$$

- Equation of line passing through point $\mathbf{a}_1$ and lying in the direction of vector $\mathbf{b}$ is

$$\mathbf{r} = \mathbf{a} + \beta \mathbf{b}$$



- **NB! Only when you** make a unit vector in the dirn of $\mathbf{b}$ does the parameter take on the length units defined by $\mathbf{a}$:

$$\mathbf{r} = \mathbf{a} + \lambda \hat{\mathbf{b}}$$

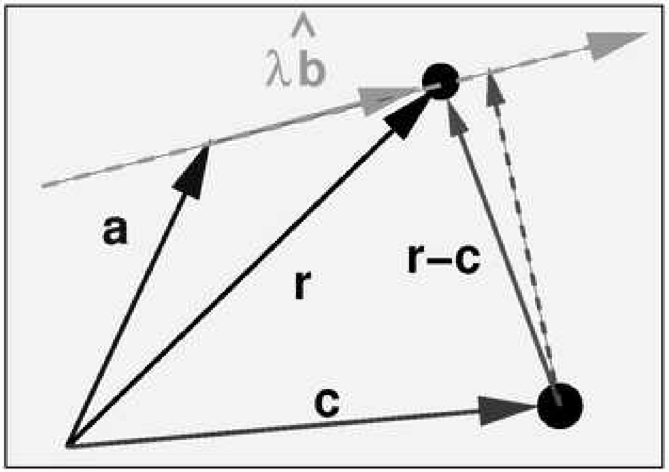
- For a line defined by two points $\mathbf{a}_1$ and $\mathbf{a}_2$

$$\mathbf{r} = \mathbf{a}_1 + \beta(\mathbf{a}_2 - \mathbf{a}_1)$$

- or the unit version ...

$$\mathbf{r} = \mathbf{a}_1 + \lambda(\mathbf{a}_2 - \mathbf{a}_1)/|\mathbf{a}_2 - \mathbf{a}_1|$$

The shortest distance from a point to a line



- Vector $\mathbf{p}$ from $\mathbf{c}$ to ANY line point $\mathbf{r}$ is

  $$\mathbf{p} = \mathbf{r} - \mathbf{c} = \mathbf{a} + \lambda\hat{\mathbf{b}} - \mathbf{c} = (\mathbf{a} - \mathbf{c}) + \lambda\hat{\mathbf{b}}$$

  which has length squared

  $$p^2 = (\mathbf{a} - \mathbf{c})^2 + \lambda^2 + 2\lambda(\mathbf{a} - \mathbf{c}) \cdot \hat{\mathbf{b}}\ .$$

- Easier to minimize p^2 rather than p itself.

  $$\frac{d}{d\lambda}p^2 = 0 \quad \text{when} \quad \lambda = -(\mathbf{a} - \mathbf{c}) \cdot \hat{\mathbf{b}}\ .$$

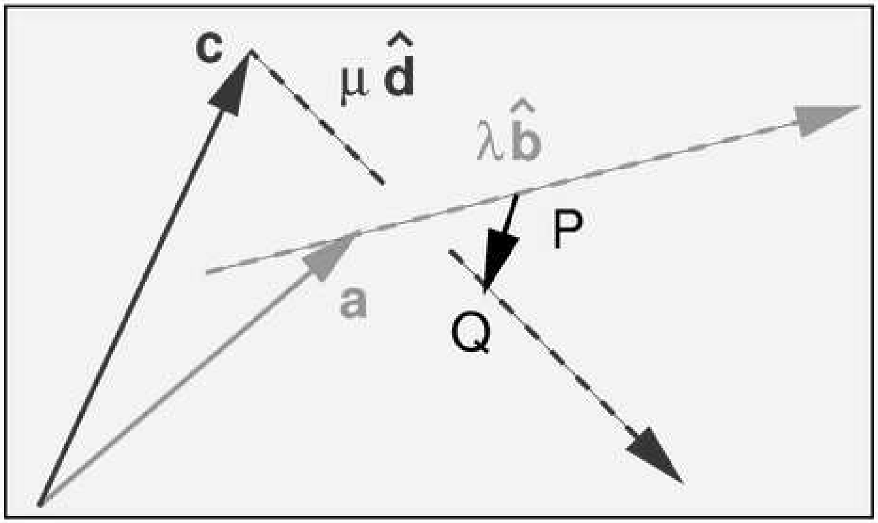
- So the minimum length vector is $\mathbf{p} = (\mathbf{a} - \mathbf{c}) - ((\mathbf{a} - \mathbf{c}) \cdot \hat{\mathbf{b}})\hat{\mathbf{b}}$.
  No surprise! It's the component of $(\mathbf{a} - \mathbf{c})$ **perpendicular** to $\hat{\mathbf{b}}$.

- We could therefore write using the "book" formula ...

  $$\begin{aligned} \mathbf{p} &= \hat{\mathbf{b}} \times [(\mathbf{a} - \mathbf{c}) \times \hat{\mathbf{b}}] \\ \Rightarrow p_{\min} &= |\hat{\mathbf{b}} \times [(\mathbf{a} - \mathbf{c}) \times \hat{\mathbf{b}}]| = |(\mathbf{a} - \mathbf{c}) \times \hat{\mathbf{b}}|\ . \end{aligned}$$

Shortest distance between two straight lines

- Shortest distance from point to line is along the perp line
- $\Rightarrow$ shortest distance between two straight lines is along mutual perpendicular.
- The lines are:
  $\mathbf{r} = \mathbf{a} + \lambda\hat{\mathbf{b}} \qquad \mathbf{r} = \mathbf{c} + \mu\hat{\mathbf{d}}$
- The unit vector along the mutual perp is



$$\hat{\mathbf{p}} = \frac{\hat{\mathbf{b}} \times \hat{\mathbf{d}}}{|\hat{\mathbf{b}} \times \hat{\mathbf{d}}|} .$$

(Yes! Don't forget that $\hat{\mathbf{b}} \times \hat{\mathbf{d}}$ is NOT a unit vector.)

- The minimum length is therefore the component of $(\mathbf{a} - \mathbf{c})$ in this direction

$$p_{\min} = \left| (\mathbf{a} - \mathbf{c}) \cdot \left(\frac{\hat{\mathbf{b}} \times \hat{\mathbf{d}}}{|\hat{\mathbf{b}} \times \hat{\mathbf{d}}|} \right) \right| .$$

♣ Example

Question
for civil engineers

Two long straight pipes are specified using Cartesian co-ordinates as follows:
Pipe A: diameter 0.8; axis through points $(2, 5, 3)$ and $(7, 10, 8)$.

Pipe B: diameter 1.0; axis through points $(0, 6, 3)$ and $(-12, 0, 9)$.

Do the pipes need re-aligning to avoid intersection?

♣ Example continued

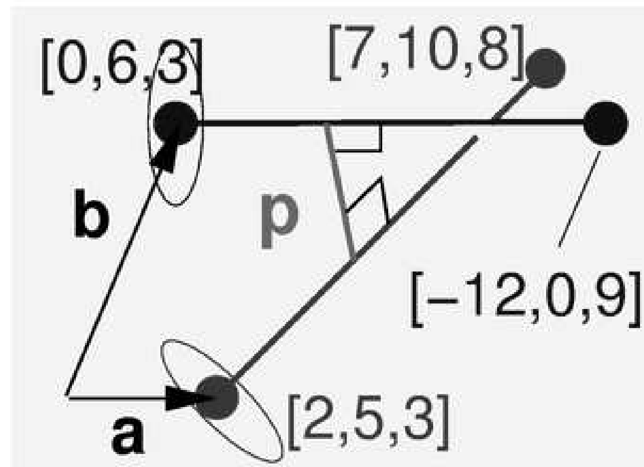
Answer

Pipes A and B have axes:

$$\mathbf{r}_A = [2, 5, 3] + \lambda'[5, 5, 5] = [2, 5, 3] + \lambda[1, 1, 1]/\sqrt{3}$$
$$\mathbf{r}_B = [0, 6, 3] + \mu'[-12, -6, 6] = [0, 6, 3] + \mu[-2, -1, 1]/\sqrt{6}$$

(Non-unit) perpendicular to both their axes is

$$\mathbf{p} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ -2 & -1 & 1 \end{vmatrix} = [2, -3, 1]$$



The length of the mutual perpendicular is mod

$$(\mathbf{a}-\mathbf{b})\cdot\frac{[2, -3, 1]}{\sqrt{14}} = [2, -1, 0]\cdot\frac{[2, -3, 1]}{\sqrt{14}} = 1.87\,.$$

Sum of the radii of the pipes is $0.4 + 0.5 = 0.9$.
Hence the pipes do not intersect.

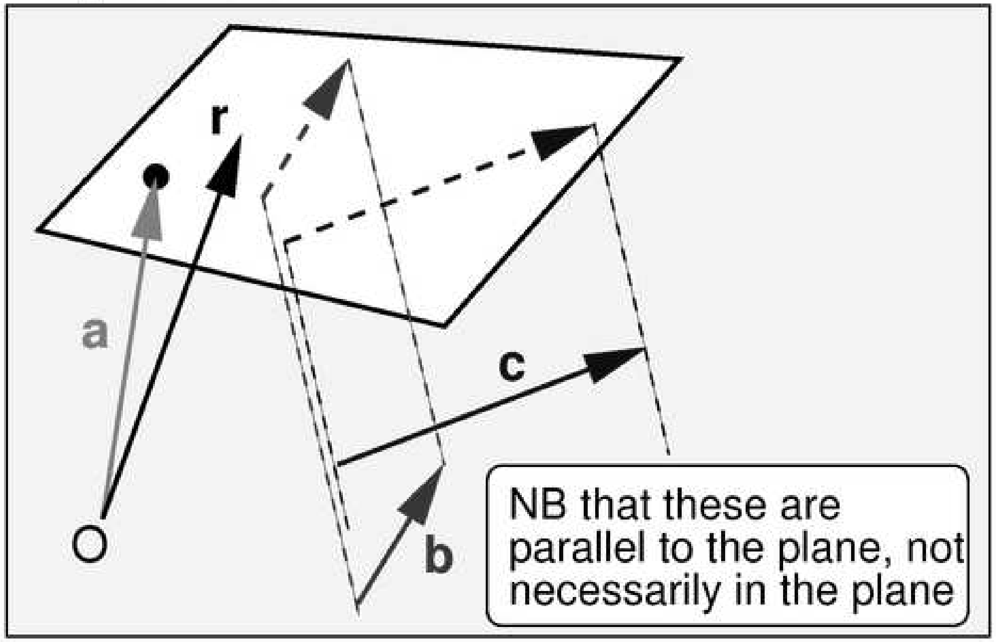
Three ways of describing a plane. Number 1

1. Point + 2 non-parallel vectors

If **b** and **c** non-parallel, and **a** is a point on the plane, then

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

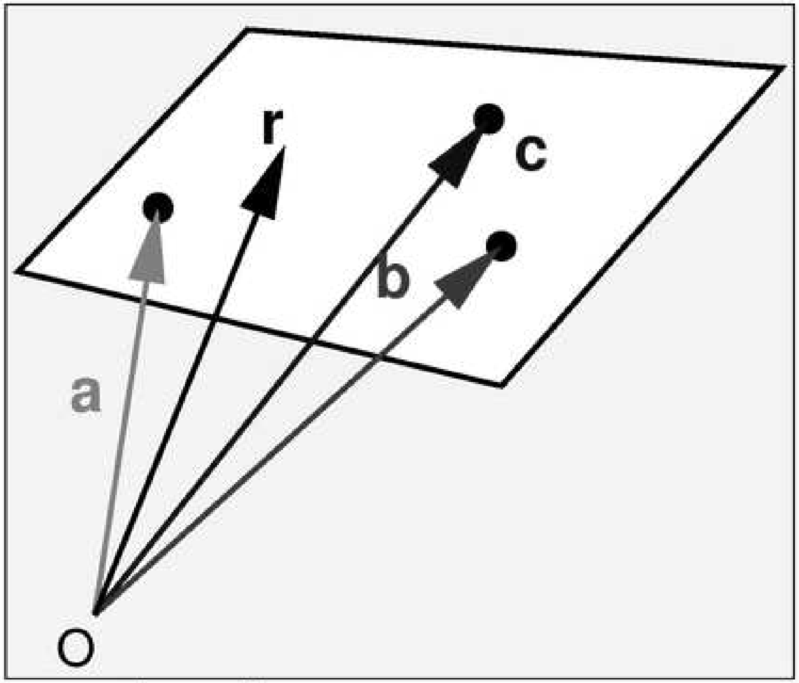
where λ, μ are scalar parameters.



2. Three points

Points **a**, **b** and **c** in the plane.

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})$$

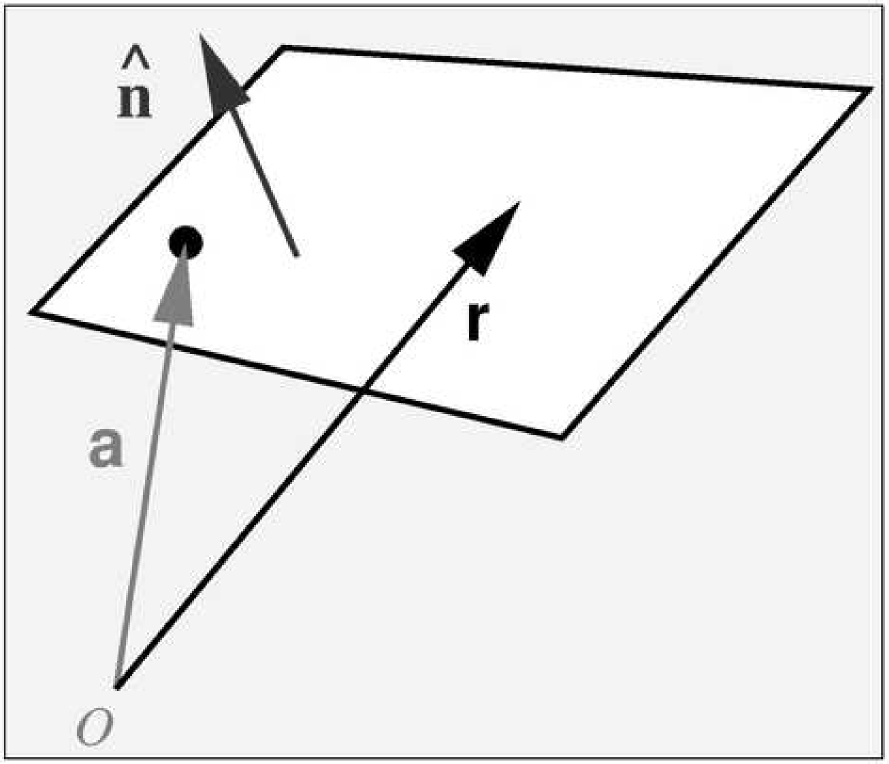


Vectors $(\mathbf{b} - \mathbf{a})$ and $(\mathbf{c} - \mathbf{a})$ are said to **span the plane**.

Three ways of describing a plane. Number 3

3. Unit normal Unit normal to the plane is $\hat{\mathbf{n}}$, and a point in the plane is $\mathbf{a}$

$$\mathbf{r} \cdot \hat{\mathbf{n}} = \mathbf{a} \cdot \hat{\mathbf{n}} = D$$



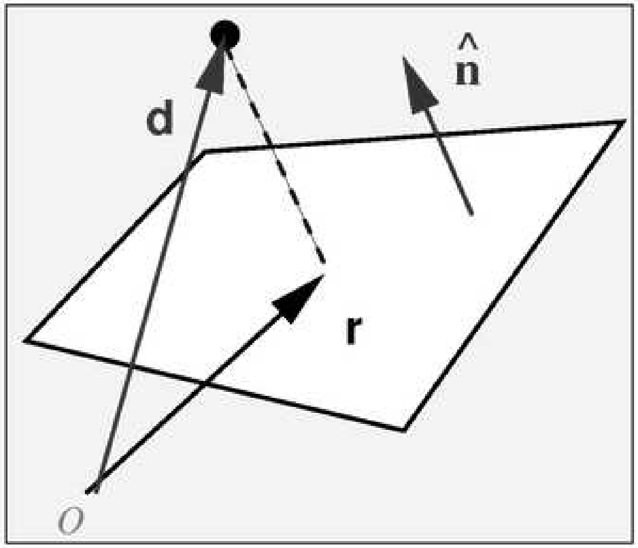
Notice that $|D|$ is the perpendicular distance to the plane from the origin. Why not just D?

- The plane is $\mathbf{r} \cdot \hat{\mathbf{n}} = \mathbf{a} \cdot \hat{\mathbf{n}} = D$
- The shortest distance d_{min} from any point to the plane is along the perpendicular.
- So, the shortest distance from the **origin** to the plane is

$$d_{\text{min}} = |D| = |\mathbf{a} \cdot \hat{\mathbf{n}}| = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{b} \times \mathbf{c}|} \quad .$$

- Now, the shortest distance from point $\mathbf{d}$ to the plane ... ?

1. Must be along the perpendicular
2. $\mathbf{d} + \lambda\hat{\mathbf{n}}$ must be a point on plane
3. $(\mathbf{d} + \lambda\hat{\mathbf{n}}) \cdot \hat{\mathbf{n}} = D$
4. $\lambda = D - \mathbf{d} \cdot \hat{\mathbf{n}}$
5. $d_{min} = |\lambda| = |D - \mathbf{d} \cdot \hat{\mathbf{n}}|$



- Find the most general vector $\mathbf{x}$ satisfying a given vector relationship.

  Eg
  $$\mathbf{x} = \mathbf{x} \times \mathbf{a} + \mathbf{b}$$

- **General Method (assuming 3 dimensions)**

  1. Set up a system of **three basis vectors** using **two** non-parallel vectors appearing in the original vector relationship. For example
  $$\mathbf{a},\ \mathbf{b},\ (\mathbf{a} \times \mathbf{b})$$
  2. Write
  $$\mathbf{x} = \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{a} \times \mathbf{b}$$
  where λ, μ, ν are scalars to be found.
  3. Substitute expression for $\mathbf{x}$ into the vector relationship to determine the set of constraints on λ,μ, and ν.

♣ Example: Solve $\mathbf{x} = \mathbf{x} \times \mathbf{a} + \mathbf{b}$.

Step 1: Basis vectors $\mathbf{a}$, $\mathbf{b}$ and v.p. $\mathbf{a} \times \mathbf{b}$.

Step 2: $\mathbf{x} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{a} \times \mathbf{b}$.

Step 3: Bung $\mathbf{x}$ back into the equation!

$$\begin{aligned} \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{a} \times \mathbf{b} &= (\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{a} \times \mathbf{b}) \times \mathbf{a} + \mathbf{b} \\ &= \mathbf{0} + \mu(\mathbf{b} \times \mathbf{a}) + \nu(\mathbf{a} \times \mathbf{b}) \times \mathbf{a} + \mathbf{b} \\ &= -\nu(\mathbf{a} \cdot \mathbf{b})\mathbf{a} + (\nu a^2 + 1)\mathbf{b} - \mu(\mathbf{a} \times \mathbf{b}) \end{aligned}$$

Equating coefficients of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ in the equation gives

$$\lambda = -\nu(\mathbf{a} \cdot \mathbf{b}) \qquad \mu = \nu a^2 + 1 \qquad \nu = -\mu$$

so that

$$\mu = \frac{1}{1 + a^2} \qquad \nu = -\frac{1}{1 + a^2} \qquad \lambda = \frac{\mathbf{a} \cdot \mathbf{b}}{1 + a^2}\ .$$

So finally the solution is the single point:

$$\mathbf{x} = \frac{1}{1 + a^2}[(\mathbf{a} \cdot \mathbf{b})\mathbf{a} + \mathbf{b} - (\mathbf{a} \times \mathbf{b})]$$

♣ Another example

Often not all the parameters are determined: μ and ν might depend on an arbitrary choice of λ (see 2A1A sheet).

And what happens if there are not two fixed vectors in the expression?

Question. Find $\mathbf{x}$ when $\mathbf{x} \cdot \mathbf{a} = K$.

Answer.

Step 1 Use $\mathbf{a}$, introduce an arbitrary vector $\mathbf{b}$, and $\mathbf{a} \times \mathbf{b}$

Step 2: $\mathbf{x} = \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{a} \times \mathbf{b}$.

Step 3: Bung $\mathbf{x}$ back into the equation!

$$\begin{aligned} \lambda a^2 + \mu \mathbf{b} \cdot \mathbf{a} &= K \\ \Rightarrow \quad \mu &= \frac{K - \lambda a^2}{\mathbf{b} \cdot \mathbf{a}} \end{aligned}$$

So, here λ, ν AND $\mathbf{b}$ are arbitary ...

$$\mathbf{x} = \lambda \mathbf{a} + \frac{K - \lambda a^2}{\mathbf{b} \cdot \mathbf{a}} \mathbf{b} + \nu \mathbf{a} \times \mathbf{b}$$

A random comment about solving vector identities

- Suppose you are faced with

$$\mu\mathbf{a} + \lambda\mathbf{b} = \mathbf{c}$$

and you want μ.

- What is the fast way of getting rid of $\mathbf{b}$?

- Use $\mathbf{b} \times \mathbf{b} = \mathbf{0}$ ...

$$\begin{aligned} \mu\mathbf{a} \times \mathbf{b} &= \mathbf{c} \times \mathbf{b} \\ \Rightarrow \mu(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) &= (\mathbf{c} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) \\ \Rightarrow \mu &= \frac{(\mathbf{c} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b})}{(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b})} \end{aligned}$$

A random comment about solving vector identities

- $\mu\mathbf{a} + \lambda\mathbf{b} = \mathbf{c}$
- An alternative is to construct two simultaneous equations

$$\begin{aligned} \mu\mathbf{a}\cdot\mathbf{b} + \lambda b^2 &= \mathbf{c}\cdot\mathbf{b} \\ \mu a^2 + \lambda\mathbf{a}\cdot\mathbf{b} &= \mathbf{a}\cdot\mathbf{c} \end{aligned}$$

and eliminate λ

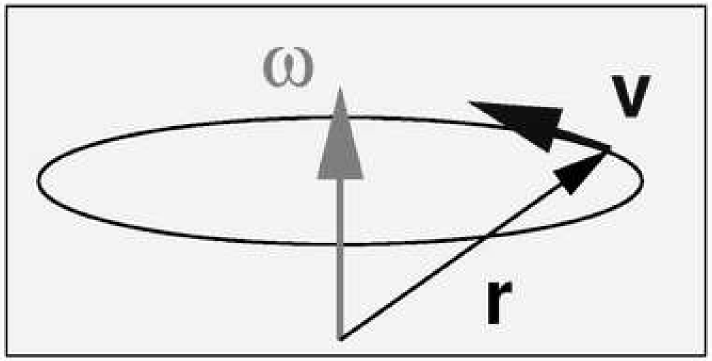
$$\mu = \frac{(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c}) - (\mathbf{a}\cdot\mathbf{c})b^2}{(\mathbf{a}\cdot\mathbf{b})^2 - a^2b^2}$$

Compare with previous

$$\mu = \frac{(\mathbf{c}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})}{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})}$$

Rotation, angular velocity and acceleration

- A rotation can represented by a vector whose
  - direction is along the axis of rotation in the sense of a right-handed screw,
  - magnitude is proportional to the size of the rotation.



- The same idea can be extended to the derivatives
  - angular velocity $\boldsymbol{\omega}$
  - angular acceleration $\dot{\boldsymbol{\omega}}$.
- The instantaneous velocity $\mathbf{v}(\mathbf{r})$ of any point P at $\mathbf{r}$ on a rigid body undergoing pure rotation can be defined by a vector product

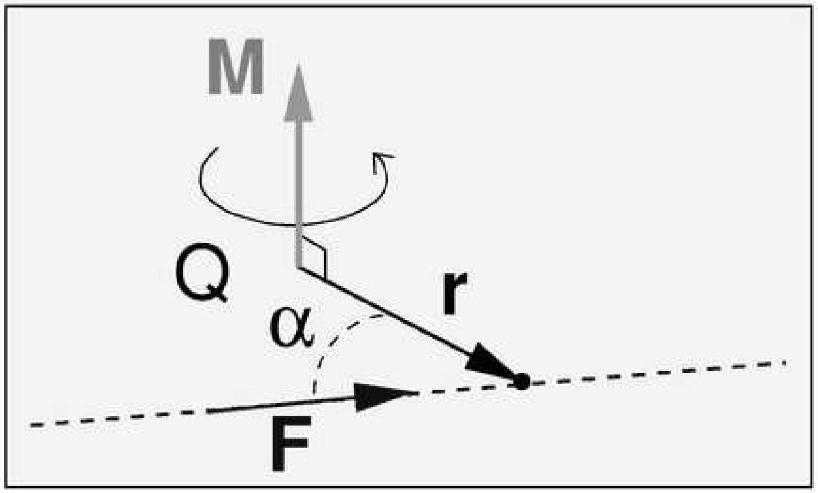
$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}.$$

Vector Moments

- Angular accelerations arise because of moments.
- The vector equation for the moment **M** of a force **F** about a point Q is

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

  where **r** is a vector from Q to *any* point on the line of action L of force **F**.



- The resulting angular acceleration $\dot{\boldsymbol{\omega}}$ is in the same direction as the moment vector **M**. (How are they related?)

Summary

Today we've discussed ...

- Vector products
- Solving vector equations
- Geometry of Lines and Planes
- Angular velocity and moments (briefly!!!)

Key point from this week:

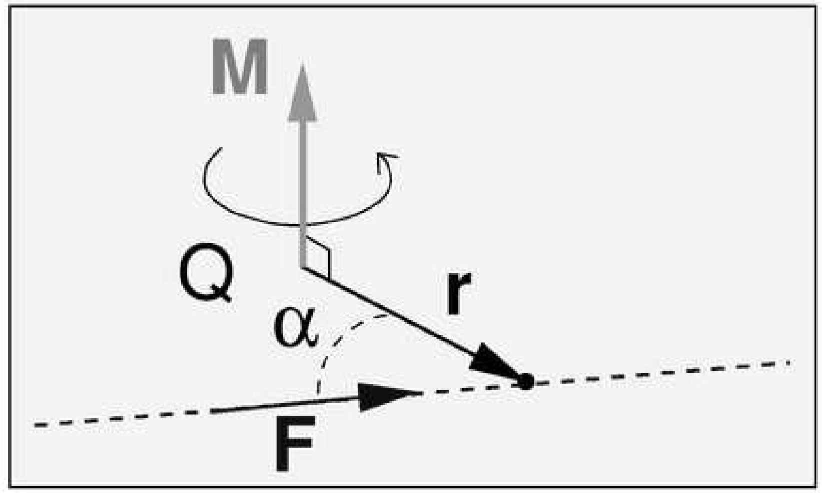
- Use vectors and their algebra "constructively" to solve problems. (The elastic collision was a good example.)
- Don't be afraid to produce solutions that involve vector operations
  Eg: $\mu = \mathbf{a} \cdot \mathbf{b} / |\mathbf{c} \times \mathbf{a}|$ Working out detail could be left to a computer program
- If you are constantly breaking vectors into their components, you are not using their power.
- Always run a consistency check that equations are vector or scalar on both sides.

Vector Moments

- Angular accelerations arise because of moments.
- The vector equation for the moment **M** of a force **F** about a point Q is

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

where **r** is a vector from Q to *any* point on the line of action L of force **F**.



- The resulting angular acceleration $\dot{\boldsymbol{\omega}}$ is in the same direction as the moment vector **M**. (How are they related?)

Summary

Today we've discussed ...

- Vector products
- Solving vector equations
- Geometry of Lines and Planes
- Angular velocity and moments (briefly!!!)

Key point from this week:

- Use vectors and their algebra "constructively" to solve problems. (The elastic collision was a good example.)
- Don't be afraid to produce solutions that involve vector operations
  Eg: $\mu = \mathbf{a} \cdot \mathbf{b} / |\mathbf{c} \times \mathbf{a}|$ Working out detail could be left to a computer program
- If you are constantly breaking vectors into their components, you are not using their power.
- Always run a consistency check that equations are vector or scalar on both sides.