Vector Algebra and Calculus

2^{nd} year A1 course

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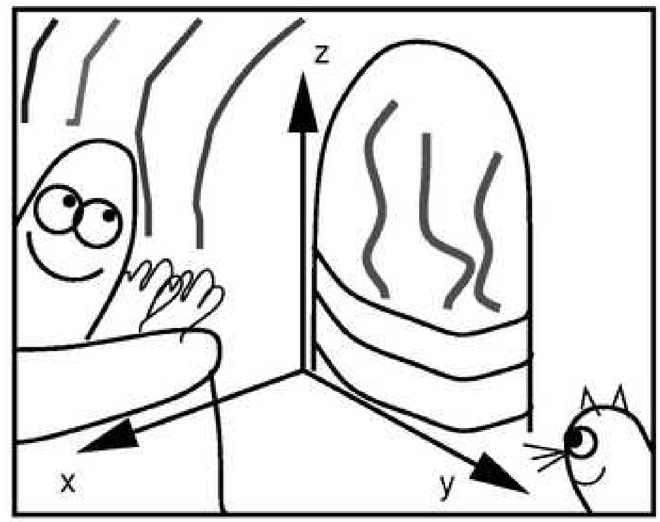
Vector Calculus & Scalar Fields

- Think about some **scalar quantities**
  — mass M, length L, time t, temperature T, etc
- If $\mathbf{r} = [x, y, z]$ is a position in space, $T(\mathbf{r})$ is a **scalar field**
- T might be time-varying — the field is $T(\mathbf{r}, t)$
- Keep y, z, t constant. What is δT when you move δx?

$$\delta T = \left(\frac{\partial T}{\partial x}\right) \delta x.$$



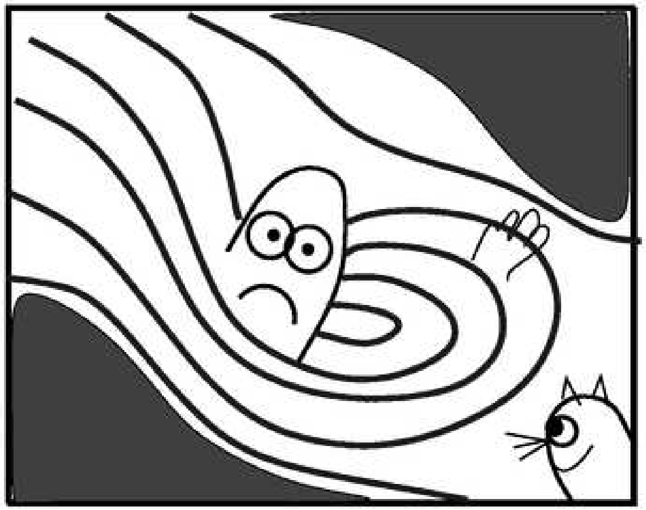
- But suppose you moved of in a direction $\mathbf{n}$. Would you multiply

$$\delta T = \frac{\partial T}{\partial \mathbf{n}} \delta \mathbf{n} \ ?$$

- Does $\partial T / \partial \mathbf{n}$ exist — is it a vector or a scalar?

Vector Calculus & Vector Fields

- A vector quantity $\mathbf{v}(\mathbf{r})$ that has a value at every $\mathbf{r}$ in a region is a **vector field**.
- Examples are:
  — The electric field $\mathbf{E}(\mathbf{r})$ around stationary charges
  — The unsteady fluid velocity field $\mathbf{v}(\mathbf{r}, t)$ in a stream.
- Local stream velocity $\mathbf{v}(\mathbf{r}, t)$ can be viewed using:
  — *laser Doppler anemometry*, or by dropping twigs in, or diving in ...



- You'll be interested in
  - weirs (acceleration), &
  - vortices (curls)

Contents

Learning Outcomes

- comfort with expressing systems using vector quantities
- manipulating vectors as "atomic" entities without recourse to underlying co-ordinates
- sound grasp of the concept of a vector field
- ability to link this idea to descriptions various physical phenomena
- intuition of the physical meaning of the various vector calculus operators (div, grad, curl)
- ability to interpret the formulae describing physical systems in terms of these operators

Reading

- J Heading, "Mathematical Methods in Science and Engineering", 2nd ed., Ch.13, (Arnold).
- G Stephenson, "Mathematical Methods for Science Students", 2nd ed., Ch.19, (Longman).
- E Kreyszig, "Advanced Engineering Mathematics", 6th ed., Ch.6, (Wiley).
- K F Riley, M. P. Hobson and S. J. Bence, "Mathematical Methods for the Physics and Engineering" Chs. 6, 8 and 9 (CUP).
- A J M Spencer, et. al. "Engineering Mathematics", Vol.1, Ch.6, (Van Nostrand Reinhold).
- H M Schey, "Div, Grad, Curl and all that", Norton

Course WWW Pages

- Pdf copies of
  - these oheads
  - lecture notes (also large print),
  - tutorial sheets (also large print)
  - FAQs etc

  will be accessible from

  www.robots.ox.ac.uk/~ian/Teaching/Vectors

- If something is *really* not clear, and you are *really* stuck,
  - email ian.reid@eng.ox.ac.uk

  and the reply (if generally useful) will get stuck on the web FAQs.

Vector Algebra and Calculus

1. Revision of vector algebra, scalar product, vector product
2. Triple products, multiple products, applications to geometry
3. Differentiation of vector functions, applications to mechanics
4. Scalar and vector fields. Line, surface and volume integrals, curvilinear co-ordinates
5. Vector operators — grad, div and curl
6. Vector Identities, curvilinear co-ordinate systems
7. Gauss' and Stokes' Theorems and extensions
8. Engineering Applications

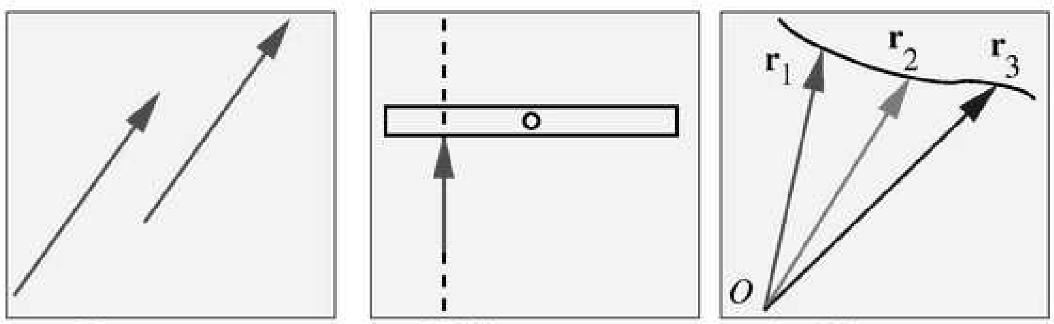
1. Vector Algebra

In which we explore ...

- Free, sliding and position vectors
- Coord frames and Vector components
- Equality, magnitude, Addition, Subtraction
- Scalar products, Vector Projection, Inner products
- Vector Products

Vectors

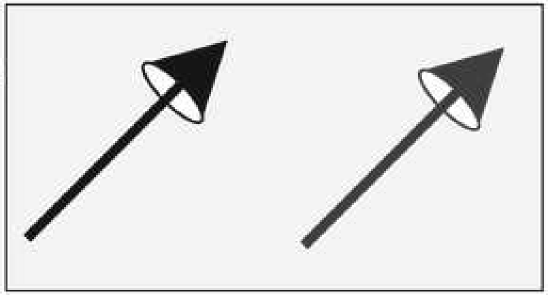
- In Linear Algebra vectors were lists of *n* numbers.
- Often in the physical world, the numbers specify
  - **magnitude** (1 number) & **direction** (1 number in 2D, 2 in 3D)
- There are three slightly different types of vectors:
  - **Free vectors:** *Only* mag & dirn are important. We can *translate* at will.
  - **Sliding vectors:** Line of action is important (eg. forces for moments) Vector can slide with 1 degree of freedom.
  - **Bound or position vectors:** "Tails" all originate at origin *O*.



Free vectors Sliding vectors Position vectors

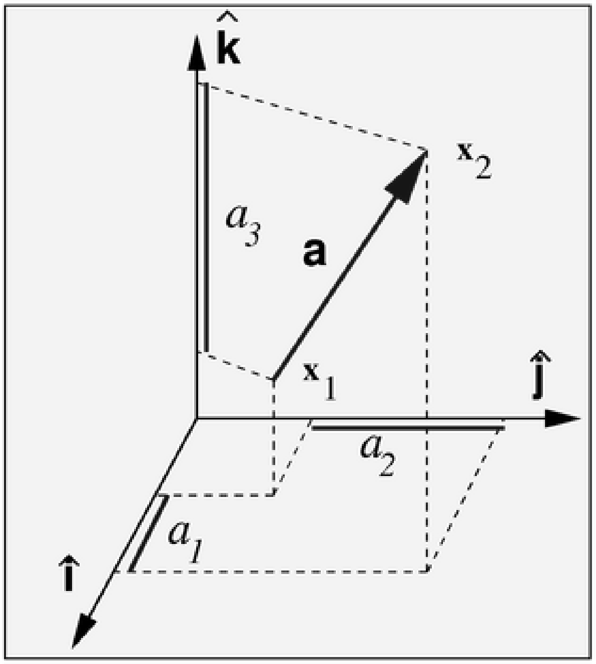
Coordinate frames

- An advantage of vector algebra:
  —frees analysis from arbitrarily imposed coordinate frames.
- Eg, two free vectors are equal if mags and dirns are equal. Can be done with a drawing that is **independent of any coordinate system**.



- Try to spot things in the notes that are independent of coordinate system.
- However, coordinate systems are useful,
  so introduce the idea of **vector components**.

Vector components in a coordinate frame



- In a Cartesian coordinate frame

$$\mathbf{a} = [a_1, a_2, a_3] = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$$

- Define $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ as unit vectors in the x, y, z dirns

$$\hat{\mathbf{i}} = [1, 0, 0] \quad \hat{\mathbf{j}} = [0, 1, 0] \quad \hat{\mathbf{k}} = [0, 0, 1]$$

then

$$\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}} \quad .$$

- Remember, general vectors not stuck in 3 dimensions!

Notation

- We will use bold font to represent vectors

$$\mathbf{a}, \boldsymbol{\omega},$$

- In written work, underline the vector

$$\underline{a}, \underline{\omega}$$

- We shall use the hat $\hat{\mathbf{a}}$ to denote a unit vector.
- $\mathbf{a}^\top$ denotes the transpose of a vector
- iff means "if and only if"
- mag and dirn are my shorthands for magnitude and direction

Vector equality

- Two free vectors are said to be equal iff their lengths and directions are the same.
- Using coordinates, two n-dimensional vectors are equal

$$\mathbf{a} = \mathbf{b} \quad \text{iff} \quad a_1 = b_1, \quad a_2 = b_2, \quad \ldots \quad a_n = b_n$$

- This does for position vectors.
- But for sliding vectors we must add *the line of action must be the same.*

Vector magnitude and unit vectors

- Provided we use an orthogonal coordinate system, the magnitude of a 3-vector is

$$a = |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

and of an *n*-vector

$$a = |\mathbf{a}| = \sqrt{\sum_i a_i^2}$$

- To find the unit vector in the direction of **a**, simply divide the vector by its magnitude

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} \ .$$

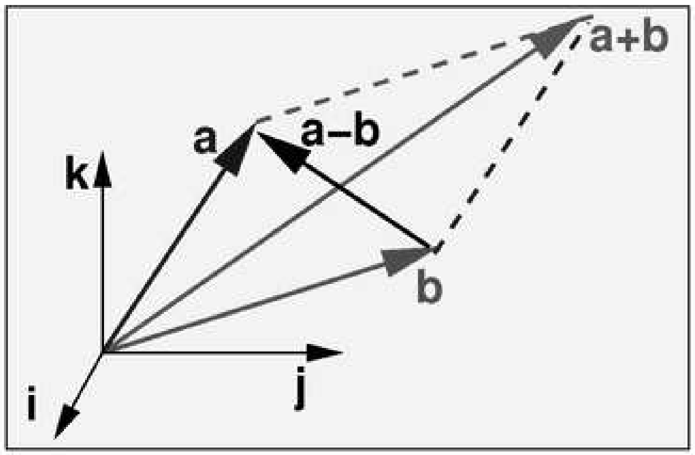
Vector Addition and Subtraction

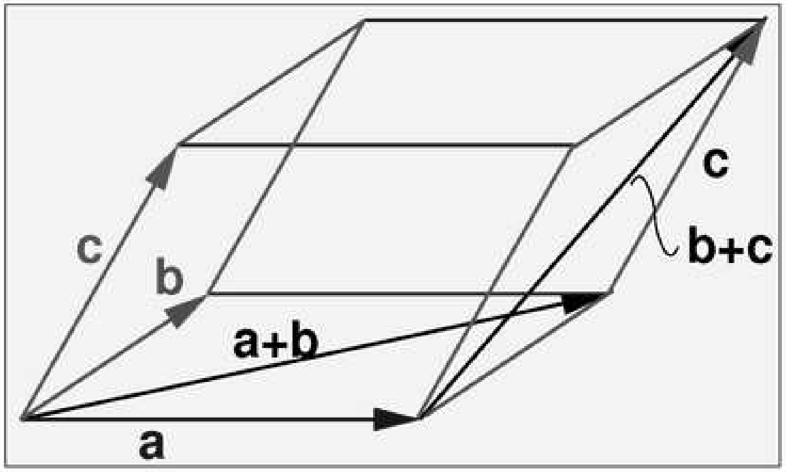
- Vectors are added/subtracted by adding/subtracting corresponding components (like matrices)

$$\mathbf{a} + \mathbf{b} = [a_1 + b_1, \quad a_2 + b_2, \quad a_3 + b_3]$$

- Addition follows the parallelogram construction.

- Subtraction is $\mathbf{a} + (-\mathbf{b})$





Properties of addition/subtraction

- The following results follow immediately from the above definition of vector addition (incl. subtraction).

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (it commutes)
2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) = \mathbf{a} + \mathbf{b} + \mathbf{c}$
   (it associates)
3. $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$
   where the zero vector is $\mathbf{0} = [0, 0, 0]$.
4. $\mathbf{a} + (\text{-}\mathbf{a}) = \mathbf{0}$

Multiplication of a vector by a scalar

- **NOT the scalar product!**
- Just as for matrices, multiplication of a vector **a** by a scalar c is defined as multiplication of each component by c, so that

$$c\mathbf{a} = [ca_1, ca_2, ca_3].$$

It follows that:

$$|c\mathbf{a}| = \sqrt{(ca_1)^2 + (ca_2)^2 + (ca_3)^2} = |c||\mathbf{a}|.$$

- The direction of the vector will reverse if c is negative, but otherwise is unaffected.
- A vector where the sign is uncertain is called a **director**.

♣ Example

- The electrostatic force on charged particle Q due to another charged particle q_1 is

$$\mathbf{F} = K\frac{Qq_1}{r^2}\hat{\mathbf{r}} \qquad \text{where constant} \quad K = \frac{1}{4\pi\epsilon_r\epsilon_0}$$

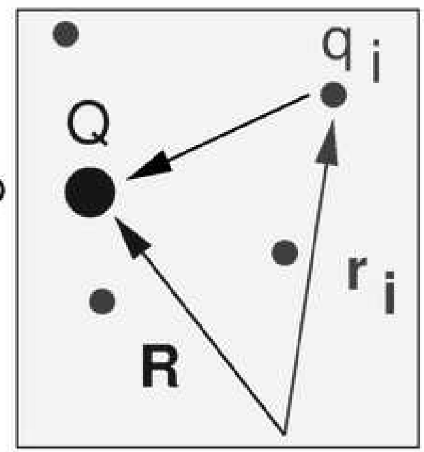
where $\mathbf{r}$ is the vector from q to Q.

Question: Write down an expression for the force on Q at $\mathbf{R}$ due to N charges q_i at $\mathbf{r}_i$, $i = 1, \ldots, N$.

Answer:

The vector from q_i to Q is $\mathbf{R} - \mathbf{r}_i$.

The unit vector in that direction is $(\mathbf{R} - \mathbf{r}_i)/|\mathbf{R} - \mathbf{r}_i|$, so



$$\mathbf{F}(\mathbf{r}) = \sum_{i=1}^{N} K\frac{Qq_i}{|\mathbf{R} - \mathbf{r}_i|^3}(\mathbf{R} - \mathbf{r}_i)$$

- Notice that we are **thinking** algebraically about **vectors** — not fussing about their components. Not a coordinate system in sight.

Scalar product or dot product

- The scalar product of two vectors results in a scalar quantity:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \; .$$

- Note that

$$\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2 = a^2.$$

- These properties of the sprod follow immediately:
  - $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
    (it commutes)
  - $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
    (it distributes w.r.t vector addition)
  - $(\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\lambda \mathbf{b})$
    (scalar multiple of a scalar product of two vectors)

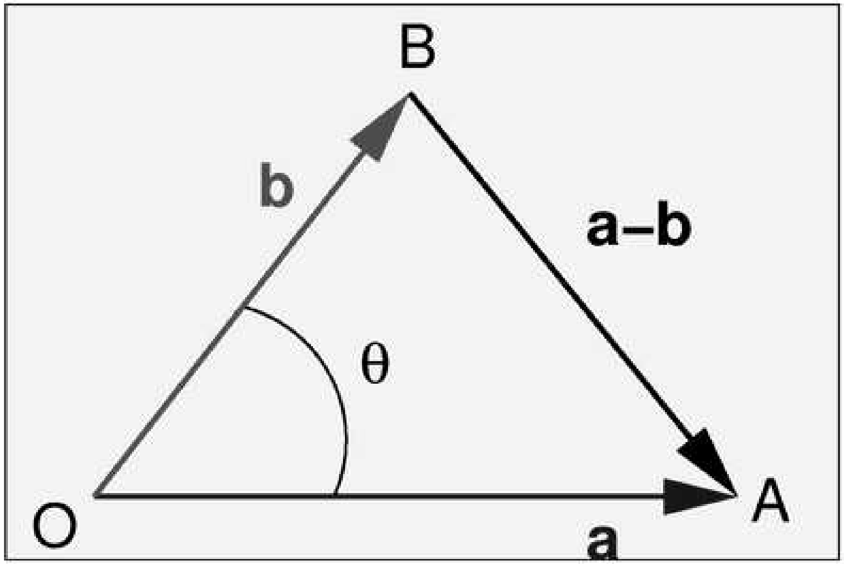
Geometrical interpretation of scalar product

- Consider the square magnitude of the vector $(\mathbf{a} - \mathbf{b})$.

$$|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2(\mathbf{a} \cdot \mathbf{b})$$
$$= a^2 + b^2 - 2(\mathbf{a} \cdot \mathbf{b})$$

- The *cosine rule* says length AB^2 is

$$|\mathbf{a} - \mathbf{b}|^2 = a^2 + b^2 - 2ab\cos\theta$$



- Hence

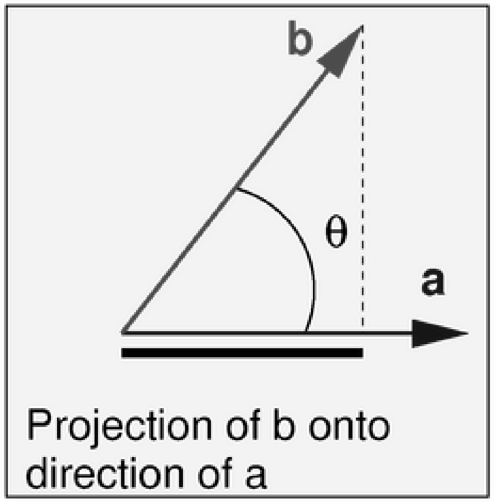
$$\mathbf{a} \cdot \mathbf{b} = ab\cos\theta,$$

independent of the coord system.

- Conversely

$$\cos\theta = \mathbf{a} \cdot \mathbf{b}/ab$$

Projection of one vector onto the other



$b \cos\theta$ is the component of **b** in the direction of **a**.
$a \cos\theta$ is the component of **a** in the direction of **b**.

- Projection is v. useful when the second vector is a unit vector.

  $\mathbf{a} \cdot \hat{\mathbf{i}}$ is the size of the component of **a** in the direction of $\hat{\mathbf{i}}$.

- To get the **vector component** of **b** in the dirn of **a**

$$(\mathbf{b} \cdot \hat{\mathbf{a}})\hat{\mathbf{a}} = \frac{1}{a^2}(\mathbf{b} \cdot \mathbf{a})\mathbf{a} \quad .$$

- So

  $(\mathbf{a} \cdot \hat{\mathbf{i}})\hat{\mathbf{i}}$ is the vector component of **a** in the direction of $\hat{\mathbf{i}}$.

Orthonormal vectors and coordinates

- In the particular case $\mathbf{a} \cdot \mathbf{b} = 0$, the angle between the two vectors is a right angle.
- The vectors are said to be orthogonal — neither has a component in the direction of the other.
- In 3D, an orthogonal coordinate system is characterised by

$$\hat{\mathbf{\imath}} \cdot \hat{\mathbf{\imath}} = \hat{\mathbf{\jmath}} \cdot \hat{\mathbf{\jmath}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$

and

$$\hat{\mathbf{\imath}} \cdot \hat{\mathbf{\jmath}} = \hat{\mathbf{\jmath}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{\imath}} = 0$$

A scalar product is an "inner product"

- We have been writing vectors as **row vectors** $\mathbf{a} = [a_1, \ a_2, \ a_3]$
- It's convenient: it takes less space than writing column vectors
- In matrix algebra, vectors are **column vectors**. So, $\mathtt{M}\mathbf{a} = \mathbf{v}$ means

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

  and row vectors are written as $\mathbf{a}^\top$ ($\mathbf{a}$ transpose).

- Most times can be relaxed, but need to fuss to point out that the scalar product is also the **inner product** used in linear algebra.
- The **inner product** is defined as $\mathbf{a}^\top\mathbf{b}$

$$\mathbf{a}^\top\mathbf{b} = [a_1, \ a_2, \ a_3] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 \ = \ \mathbf{a} \cdot \mathbf{b}$$

♣ Scalar Product Example 1

Question

A force $\mathbf{F}$ is applied to an object as it moves by a small amount $\delta\mathbf{r}$.
What work is done on the object by the force?

Answer

The work done is equal to the component of force in the direction of the displacement multiplied by the displacement itself.
This is just a scalar product:

$$\delta W = \mathbf{F} \cdot \delta\mathbf{r} \quad .$$

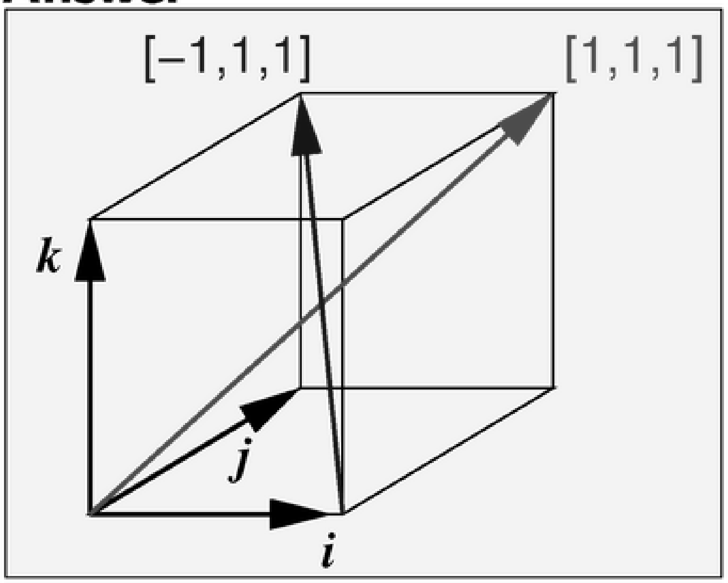
Later we will see how to integrate such elements over particular paths as *line integrals*.

♣ Scalar Product Example 2

Question

A cube has four diagonals, connecting opposite vertices. What is the angle between an adjacent pair?

Answer



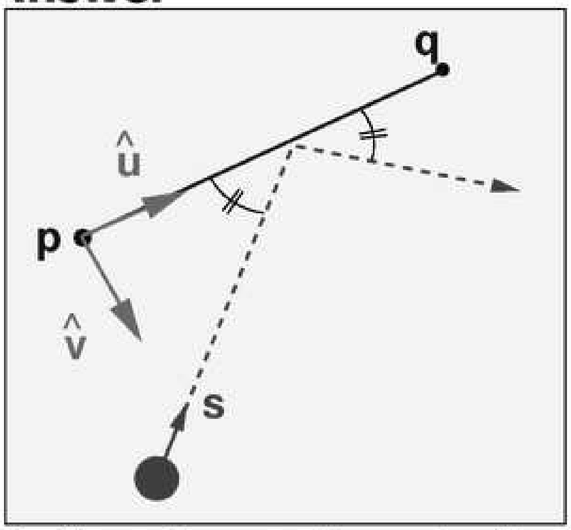
The directions of the diagonals are $[\pm1, \pm1, \pm1]$. The ones shown in the figure are $[1, 1, 1]$ and $[-1, 1, 1]$. The angle is thus

$$\begin{aligned}\theta &= \cos^{-1}\frac{[1,1,1]\cdot[-1,1,1]}{\sqrt{1^2+1^2+1^2}\sqrt{-1^2+1^2+1^2}}\\ &= \cos^{-1}(1/3)\end{aligned}$$

♣ Scalar Product Example 3

Question: Pinball with velocity $\mathbf{s}$ bounces (elastically) from a baffle whose end-points are $\mathbf{p}$ and $\mathbf{q}$. What is the velocity vector after the bounce?

Answer



Refer to coord frame with principal directions along and perpendicular to the baffle:

$$\hat{\mathbf{u}} = [u_x, u_y] = \frac{\mathbf{q}-\mathbf{p}}{|\mathbf{q}-\mathbf{p}|}$$

$$\hat{\mathbf{v}} = \mathbf{u}^{\perp} = [-u_y, \quad u_x]$$

Before impact: velocity is $\mathbf{s}_{\text{before}} = (\mathbf{s}.\hat{\mathbf{u}})\hat{\mathbf{u}} + (\mathbf{s}.\hat{\mathbf{v}})\hat{\mathbf{v}}$

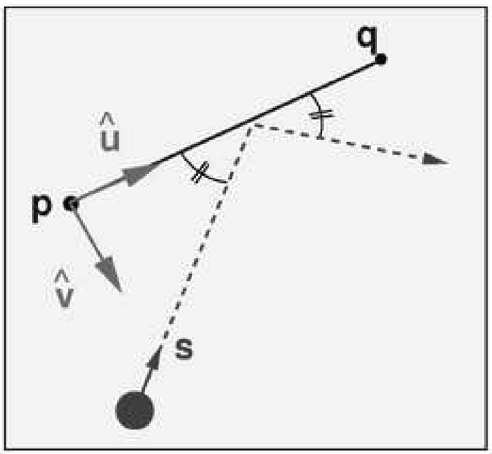
After impact: component of velocity in dirn of baffle $\hat{\mathbf{u}}$ is **same**

. component normal to the baffle along $\hat{\mathbf{v}}$ is **reversed**

$$\Rightarrow \quad \mathbf{s}_{\text{after}} = (\mathbf{s}.\hat{\mathbf{u}})\hat{\mathbf{u}} - (\mathbf{s}.\hat{\mathbf{v}})\hat{\mathbf{v}}$$

♣ Scalar Product Example 3

- Worth reflecting on this example ...
- Using vectors as complete entities (ie, not thinking about components) has made a tricky problem trivial to solve.
- Several languages (including Matlab) allow one to declare vector objects




```
p=[3;4]
q=[1;-1]
s=[1;2]
diff = q-p
uhat = diff/norm(diff)
vhat = [-uhat(2);uhat(1)]
safter = dot(s,uhat)*uhat - dot(s,vhat)*vhat
```

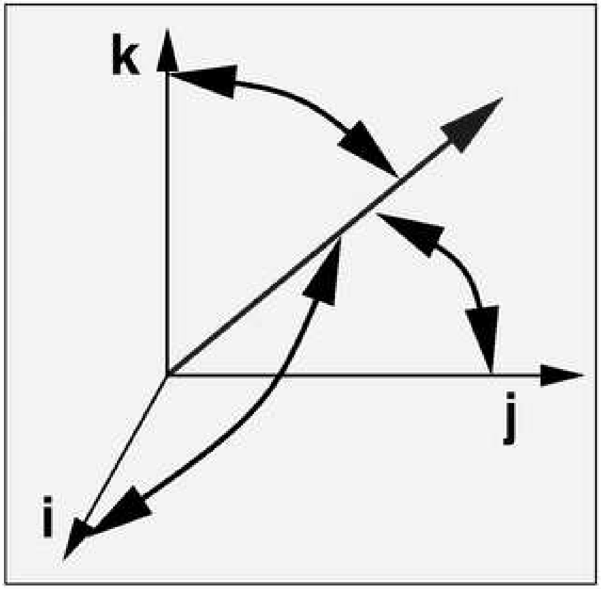
- You think in vectors, while built in routines handle the detail of components
- ... Reflection over.

Direction cosines use projection

- The quantities

$$\lambda = \frac{\mathbf{a}\cdot\hat{\mathbf{\imath}}}{a}, \qquad \mu = \frac{\mathbf{a}\cdot\hat{\mathbf{\jmath}}}{a}, \qquad \nu = \frac{\mathbf{a}\cdot\hat{\mathbf{k}}}{a}$$

are the cosines of the angles which the vector **a** makes with the coordinate vectors $\hat{\mathbf{\imath}}$,$\hat{\mathbf{\jmath}}$, $\hat{\mathbf{k}}$



- They are the **direction cosines** of the vector **a**.
- Since $\mathbf{a}\cdot\hat{\mathbf{\imath}} = a_1$ etc, it follows immediately that

$$\mathbf{a} = a(\lambda\hat{\mathbf{\imath}} + \mu\hat{\mathbf{\jmath}} + \nu\hat{\mathbf{k}})$$

$$\lambda^2 + \mu^2 + \nu^2 = \frac{1}{a^2}[a_1^2 + a_2^2 + a_3^2] = 1$$

Vector or cross product

- The **vector product** of two vectors **a** and **b** is

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\hat{\mathbf{\imath}} + (a_3 b_1 - a_1 b_3)\hat{\mathbf{\jmath}} + (a_1 b_2 - a_2 b_1)\hat{\mathbf{k}}.$$

- You *cannot* remember the above! Instead use the **pseudo determinant**

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where the top row consists of vectors not scalars.

- A determinant with two equal rows has value zero, so

$$\mathbf{a} \times \mathbf{a} = 0$$

- It is also easily verified that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$$

so that $\mathbf{a} \times \mathbf{b}$ is **orthogonal** to both **a** and **b**.

Vector product

- The magnitude of the vector product can be obtained by showing that

$$|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = a^2 b^2$$

from which it follows (independent of the coord system)

$$|\mathbf{a} \times \mathbf{b}| = ab \sin\theta \ ,$$

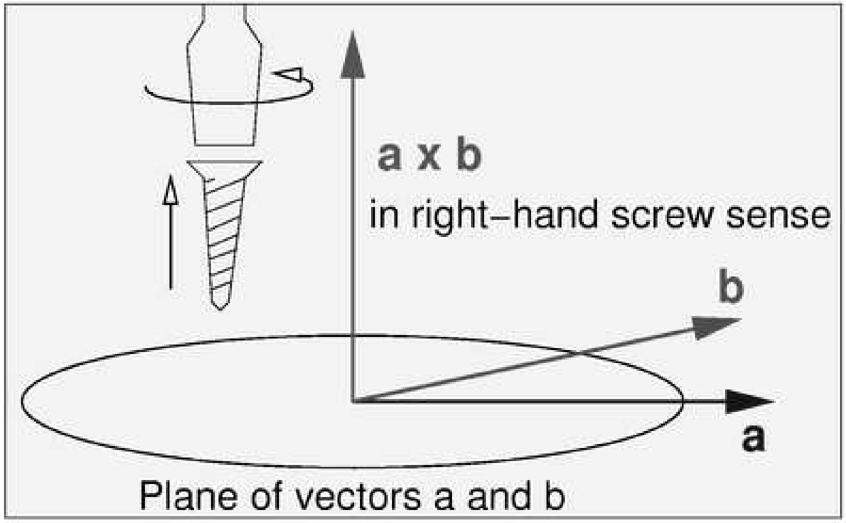
- Proof?

- The vector product does not commute
It **anti-commutes**: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

- The vector product does not associate:
$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

Vector Products

- The vector product is orthogonal to both the vectors.
- Need to specify the sense w.r.t these vectors.
- Sense of the right handed screw ...



- Also

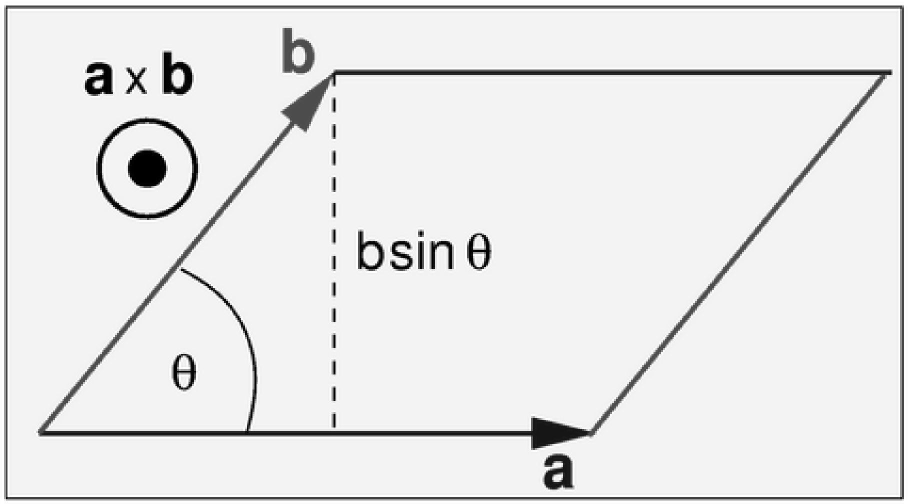
$$\hat{\mathbf{\imath}} \times \hat{\mathbf{\jmath}} = \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \hat{\mathbf{k}} \quad .$$

- And in full: $\hat{\mathbf{\imath}} \times \hat{\mathbf{\jmath}} = \hat{\mathbf{k}}$, $\hat{\mathbf{\jmath}} \times \hat{\mathbf{k}} = \hat{\mathbf{\imath}}$, and $\hat{\mathbf{k}} \times \hat{\mathbf{\imath}} = \hat{\mathbf{\jmath}}$.

  Note the cycle ordering here.

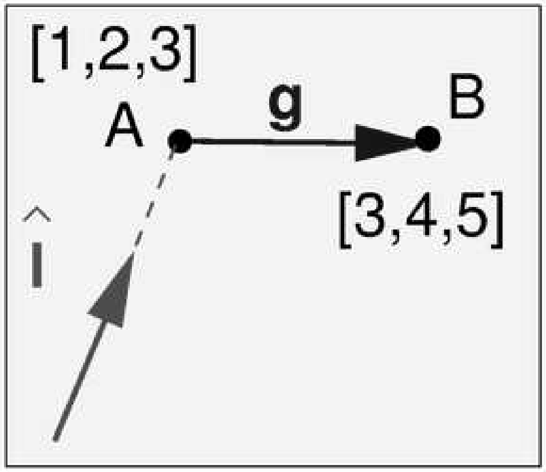
Geometrical interpretation of vector product

- The magnitude of the vector product ($\mathbf{a} \times \mathbf{b}$) is equal to the area of the parallelogram whose sides are parallel to, and have lengths equal to the magnitudes of, the vectors **a** and **b**.
- Its direction is perpendicular to the parallelogram.



Question

$\mathbf{g}$ is vector from A [1,2,3] to B [3,4,5].
$\hat{\mathbf{l}}$ is the unit vector in dirn from O to A.
Find $\hat{\mathbf{m}}$, a UNIT vector along $\mathbf{g} \times \hat{\mathbf{l}}$
Verify that $\hat{\mathbf{m}}$ is is perpendicular to $\hat{\mathbf{l}}$.
Find $\hat{\mathbf{n}}$, the third member of a r-h coord set $\hat{\mathbf{l}}, \hat{\mathbf{m}}, \hat{\mathbf{n}}$.



Answer

1) $\mathbf{g} = [3-1, 4-2, 5-3] = [2,2,2]$.

2) $\hat{\mathbf{l}} = [1,2,3]/\sqrt{14}$

3) $\mathbf{g} \times \mathbf{l} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = [2, -4, 2]$

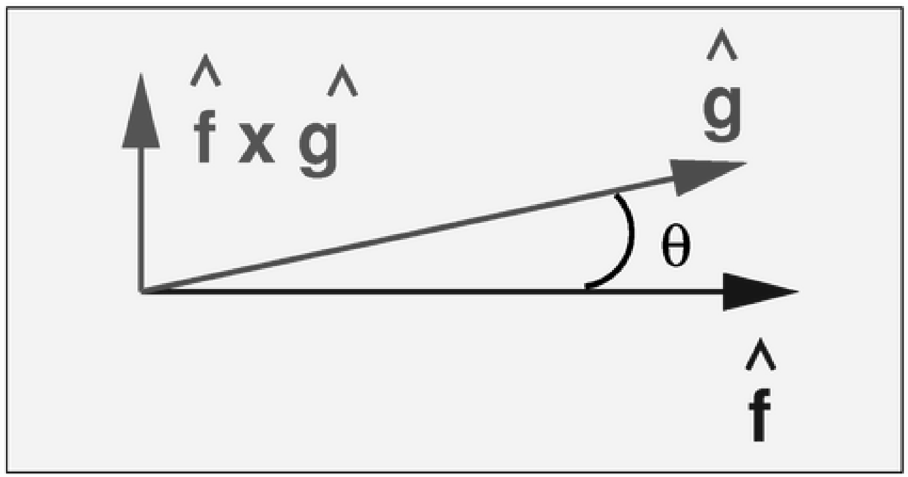
$\Rightarrow \hat{\mathbf{m}} = [1,-2,1]/\sqrt{6}$

4) $\hat{\mathbf{l}} \cdot \hat{\mathbf{m}} = (1.1 + 2. -2 + 1.3)/(.) = 0$

5) $\hat{\mathbf{n}} = \hat{\mathbf{l}} \times \hat{\mathbf{m}} = \frac{1}{\sqrt{6}\sqrt{14}} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 3 \\ 1 & -2 & 1 \end{vmatrix}$

Q: If $\hat{\mathbf{f}}$ and $\hat{\mathbf{g}}$ are two unit vectors, what is the magnitude of the vector product

$$\hat{\mathbf{f}} \times \hat{\mathbf{g}}$$



A: Magnitude is $\sin\theta$.

Summary

We've revised and discussed ...

- Free, sliding and position vectors
- Coord frames and Vector components
- Equality, magnitude, Addition, Subtraction
- Scalar products, Vector Projection, Inner products
- Vector Products

In Lecture 2 ...

- Vector multiple products:
- Geometry of Lines and Planes
- Solving vector equations
- Angular velocity and moments

Then the calculus starts