Lecture 4

Line, Surface and Volume Integrals. Curvilinear coordinates.

We started off the course being concerned with individual vectors **a**, **b**, **c**, and so on.

We went on to consider how single vectors vary over time or over some other parameter such as arc length.

In much of the rest of the course, we will be concerned with scalars and vectors which are defined over regions in space — scalar and vector *fields*

In this lecture we introduce line, surface and volume integrals, and consider how these are defined in non-Cartesian, *curvilinear coordinates*

4.1 Scalar and vector fields

When a scalar function $u(\mathbf{r})$ is determined or defined at each position $\mathbf{r}$ in some region, we say that u is a **scalar field** in that region.

Similarly, if a vector function $\mathbf{v}(\mathbf{r})$ is defined at each point, then $\mathbf{v}$ is a **vector field** in that region. As you will see, in **field theory** our aim is to derive statements about the bulk properties of scalar and vector fields, rather than to deal with individual scalars or vectors.

Familiar examples of each are shown in figure 4.1.

In Lecture 1 we worked out the force $\mathbf{F}(\mathbf{r})$ on a charge Q arising from a numbers of charges q_i. The electric field is $\mathbf{F}/Q$, so

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{N} K \frac{q_i}{|\mathbf{r}-\mathbf{r}_i|^3}(\mathbf{r}-\mathbf{r}_i) \quad . \qquad (K = \frac{1}{4\pi\epsilon_r\epsilon_0})$$

You could work out the velocity field in plane polars at any point on a wheel

(a) (b)

Figure 4.1: Examples of (a) a scalar field (pressure); (b) avector field (wind velocity)

spinning about its axis

$$\mathbf{v}(\mathbf{r}) = \boldsymbol{\omega} \times \mathbf{r}$$

or the fluid flow field around a wing.

If the fields are independent of time, they are said to be *steady*. Of course, most vector fields of practical interest in engineering science are not steady, and some are unpredictable.

Let us first consider how to perform a variety of types of integration in vector and scalar fields.

4.2 Line integrals through fields

Line integrals are concerned with measuring the integrated interaction with a field as you move through it on some defined path. Eg, given a map showing the pollution density field in Oxford, you may wish to work out how much pollution you breath in when cycling from college to the Department via different routes.

First recall the definition of an integral for a scalar function $f(x)$ of a single scalar variable x. One assumes a set of n samples $f_i = f(x_i)$ spaced by δx_i. One forms the limit of the sum of the products $f(x_i)\delta x_i$ as the number of samples tends to infinity

$$\int f(x)dx = \lim_{\substack{n \to \infty \\ \delta x_i \to 0}} \sum_{i=1}^{n} f_i \delta x_i \quad .$$

For a smooth function, it is irrelevant how the function is subdivided.

4.2.1 Vector line integrals

In a vector line integral, the path L along which the integral is to be evaluated is split into a large number of *vector* segments $\delta\mathbf{r}_i$. Each line segment is then

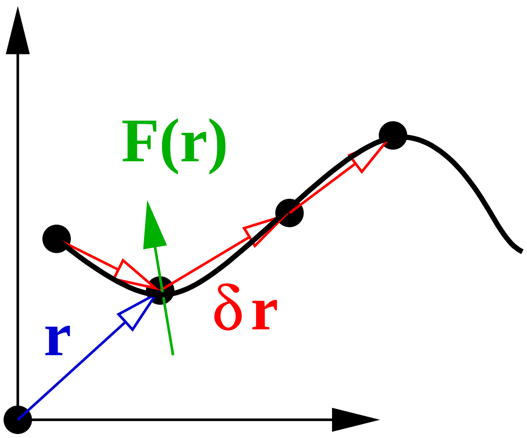


Figure 4.2: Line integral. In the diagram $\mathbf{F}(\mathbf{r})$ is a vector field, but it could be replace with scalar field $U(\mathbf{r})$.

multiplied by the quantity associated with that point in space, the products are then summed and the limit taken as the lengths of the segments tend to zero.

There are three types of integral we have to think about, depending on the nature of the product:

1. Integrand $U(\mathbf{r})$ is a scalar field, hence the integral is a vector.

$$\mathbf{I} = \int_L U(\mathbf{r}) d\mathbf{r} \qquad \left(= \lim_{\delta \mathbf{r}_i \to 0} \sum_i U_i \boldsymbol{\delta} \mathbf{r}_i . \right)$$

2. Integrand $\mathbf{a}(\mathbf{r})$ is a vector field dotted with $d\mathbf{r}$ hence the integral is a scalar:

$$I = \int_L \mathbf{a}(\mathbf{r}) \cdot d\mathbf{r} \qquad \left(= \lim_{\delta \mathbf{r}_i \to 0} \sum_i \mathbf{a}_i \cdot \boldsymbol{\delta} \mathbf{r}_i . \right)$$

3. Integrand $\mathbf{a}(\mathbf{r})$ is a vector field crossed with $d\mathbf{r}$ hence vector result.

$$\mathbf{I} = \int_L \mathbf{a}(\mathbf{r}) \times d\mathbf{r} \qquad \left(= \lim_{\delta \mathbf{r}_i \to 0} \sum_i \mathbf{a}_i \times \boldsymbol{\delta} \mathbf{r}_i . \right)$$

Note immediately that unlike an integral in a single scalar variable, there are many paths L from start point $\mathbf{r}_A$ to end point $\mathbf{r}_B$, and the integral will in general depend on the path taken.

Physical examples of line integrals

- The total work done by a force $\mathbf{F}$ as it moves a point from A to B along a given path C is given by a line integral of type 2 above. If the force acts at point $\mathbf{r}$ and the instantaneous displacement along curve C is $d\mathbf{r}$ then the infinitessimal work done is $dW = \mathbf{F}.d\mathbf{r}$, and so the total work done traversing the path is

$$W_C = \int_C \mathbf{F}.d\mathbf{r}$$

- Ampère's law relating magnetic field $\mathbf{B}$ to linked current can be written as

$$\oint_C \mathbf{B}.d\mathbf{r} = \mu_0 I$$

where I is the current enclosed by (closed) path C.

- The force on an element of wire carrying current I, placed in a magnetic field of strength $\mathbf{B}$, is $d\mathbf{F} = I d\mathbf{r} \times \mathbf{B}$. So if a loop this wire C is placed in the field then the total force will be and integral of type 3 above:

$$\mathbf{F} = I \oint_C d\mathbf{r} \times \mathbf{B}$$

Note that the expressions above are beautifully compact in vector notation, and are all independent of coordinate system. Of course when evaluating them we need to choose a coordinate system: often this is the standard Cartesian coordinate system (as in the worked examples below), but need not be, as we shall see in section 4.6.

♣ Examples

Q1 An example in the xy-plane. A force $\mathbf{F} = x^2y\hat{\boldsymbol{\imath}} + xy^2\hat{\boldsymbol{\jmath}}$ acts on a body at it moves between $(0, 0)$ and $(1, 1)$.

Determine the work done when the path is

1. along the line $y = x$.
2. along the curve $y = x^n$.
3. along the x axis to the point $(1, 0)$ and then along the line $x = 1$

A1 This is an example of the "type 2" line integral. In planar Cartesians, $d\mathbf{r} = \hat{\boldsymbol{\imath}}dx + \hat{\boldsymbol{\jmath}}dy$. Then the work done is

$$\int_L \mathbf{F} \cdot d\mathbf{r} = \int_L (x^2ydx + xy^2dy) \ .$$

1. For the path $y = x$ we find that $dy = dx$. So it is easiest to convert all y references to x.

$$\int_{(0,0)}^{(1,1)} (x^2ydx+xy^2dy) = \int_{x=0}^{x=1} (x^2xdx+xx^2dx) = \int_{x=0}^{x=1} 2x^3dx = \left[x^4/2\right|_{x=0}^{x=1} = 1/2 \ .$$

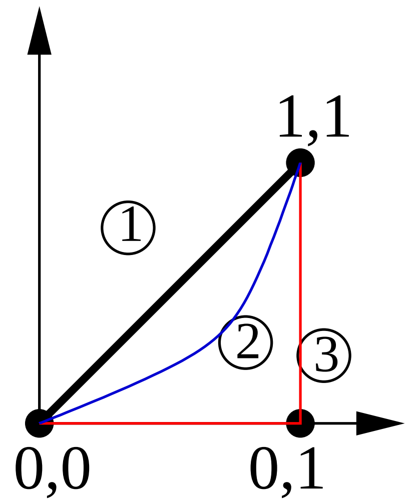


Figure 4.3: Line integral taken along three difference paths

2. For the path $y = x^n$ we find that $dy = nx^{n-1}dx$, so again it is easiest to convert all y references to x.

$$\begin{aligned}\int_{(0,0)}^{(1,1)} (x^2y\,dx + xy^2dy) &= \int_{x=0}^{x=1} (x^{n+2}dx + nx^{n-1}.x.x^{2n}dx) \\ &= \int_{x=0}^{x=1} (x^{n+2}dx + nx^{3n}dx) \\ &= \frac{1}{n+3} + \frac{n}{3n+1}\end{aligned}$$

3. This path is not smooth, so break it into two. Along the first section, $y = 0$ and $dy = 0$, and on the second $x = 1$ and $dx = 0$, so

$$\int_A^B (x^2y\,dx + xy^2dy) = \int_{x=0}^{x=1} (x^2 0\,dx) + \int_{y=0}^{y=1} 1.y^2dy = 0 + \left[y^3/3\right|_{y=0}^{y=1} = 1/3\ .$$

So in general the integral depends on the path taken. Notice that answer (1) is the same as answer (2) when $n = 1$, and that answer (3) is the limiting value of answer (2) as $n \to \infty$.

Q2 Repeat part (2) using the Force $\mathbf{F} = xy^2\hat{\boldsymbol{\imath}} + x^2y\hat{\boldsymbol{\jmath}}$.

A2 For the path $y = x^n$ we find that $dy = nx^{n-1}dx$, so

$$\begin{aligned}\int_{(0,0)}^{(1,1)} (y^2 x dx + yx^2 dy) &= \int_{x=0}^{x=1} (x^{2n+1}dx + nx^{n-1}.x^2.x^n dx) \\ &= \int_{x=0}^{x=1} (x^{2n+1}dx + nx^{2n+1}dx) \\ &= \frac{1}{2n+2} + \frac{n}{2n+2} \\ &= \frac{1}{2} \quad \text{independent of } n\end{aligned}$$

4.3 Line integrals in Conservative fields

In the second example, the line integral has the same value for the whole range of paths. In fact it is wholly independant of path. This is easy to see if we write $g(x, y) = x^2y^2/2$. Then using the definition of the perfect differential

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy$$

we find that

$$\begin{aligned}\int_A^B (y^2 x dx + yx^2 dy) &= \int_A^B dg \\ &= g_B - g_A\end{aligned}$$

which depends solely on the value of g at the start and end points, and not at all on the path used to get from A to B. Such a vector field is called **conservative**.

One sort of line integral performs the integration around a complete loop and is denoted with a ring. If **E** is a conservative field, determine the value of

$$\oint \mathbf{E} \cdot d\mathbf{r} \ .$$

In electrostatics, if **E** is the electric field the the potential function is

$$\phi = -\int \mathbf{E} \cdot d\mathbf{r} \ .$$

Do you think **E** is conservative?

4.3.1 A note on line integrals defined in terms of arc length

Line integrals are often defined in terms of scalar arc length. They don't appear to involve vectors (but actually they are another form of type 2 defined earlier).

The integrals usually appears as follows

$$I = \int_L F(x, y, z) ds$$

and most often the path L is along a curve defined parametrically as $x = x(p)$, $y = y(p)$, $z = z(p)$ where p is some parameter. Convert the function to $F(p)$, writing

$$I = \int_{p_{\text{start}}}^{p_{\text{end}}} F(p) \frac{ds}{dp} \, dp$$

where

$$\frac{ds}{dp} = \left[\left(\frac{dx}{dp} \right)^2 + \left(\frac{dy}{dp} \right)^2 + \left(\frac{dz}{dp} \right)^2 \right]^{1/2} \quad .$$

Note that the parameter p could be arc-length s itself, in which case $ds/dp = 1$ of course! Another possibility is that the parameter p is x — that is we are told $y = y(x)$ and $z = z(x)$. Then

$$I = \int_{x_{\text{start}}}^{x_{\text{end}}} F(x) \left[1 + \left(\frac{dy}{dx} \right)^2 + \left(\frac{dz}{dx} \right)^2 \right]^{1/2} \, dx \quad .$$

4.4 Surface integrals

These can be defined by analogy with line integrals.

The surface S over which the integral is to be evaluated is now divided into infinitesimal vector elements of area $d\mathbf{S}$, the direction of the vector $d\mathbf{S}$ representing the direction of the surface normal and its magnitude representing the area of the element.

Again there are three possibilities:

- $\int_S U d\mathbf{S}$ — scalar field U; vector integral.
- $\int_S \mathbf{a} \cdot d\mathbf{S}$ — vector field $\mathbf{a}$; scalar integral.
- $\int_S \mathbf{a} \times d\mathbf{S}$ — vector field $\mathbf{a}$; vector integral.

(in addition, of course, to the purely scalar form, $\int_S U dS$).

Physical example of surface integral

- Physical examples of surface integrals with vectors often involve the idea of *flux* of a vector field through a surface, $\int_S \mathbf{a}.d\mathbf{S}$ For example the mass of fluid crossing a surface S in time dt is $dM = \rho\mathbf{v}.d\mathbf{S}dt$ where $\rho(\mathbf{r})$ is the fluid density and $\mathbf{v}(\mathbf{r})$ is the fluid velocity. The total mass flux can be expressed as a surface integral:

$$\Phi_M = \int_S \rho(r)\mathbf{v}(\mathbf{r}).s\mathbf{S}$$

Again, though this expression is coordinate free, we evaluate an example below using Cartesians. Note, however, that in some problems, symmetry may lead us to a different more natural coordinate system.

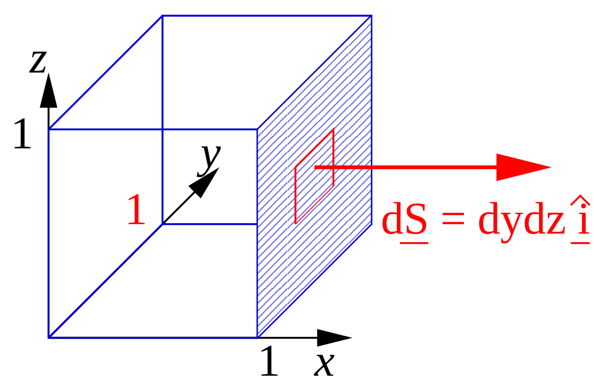
♣ **Example**

Evaluate $\int \mathbf{F}\cdot d\mathbf{S}$ over the $x = 1$ side of the cube shown in the figure when $\mathbf{F} = y\hat{\boldsymbol{\imath}} + z\hat{\boldsymbol{\jmath}} + x\hat{\boldsymbol{k}}$.
$d\mathbf{S}$ is perpendicular to the surface. Its $\pm$ direction actually depends on the nature of the problem. More often than not, the surface will enclose a volume, and the surface direction is taken as everywhere emanating from the interior.
Hence for the $x = 1$ face of the cube

$$d\mathbf{S} = dydz\hat{\boldsymbol{\imath}}$$

and

$$\begin{aligned}\int \mathbf{F}\cdot d\mathbf{S} &= \int\int ydydz \\ &= \frac{1}{2}y^2\Big|_0^1 z\Big|_0^1 = \frac{1}{2} \quad .\end{aligned}$$



4.5 Volume integrals

The definition of the volume integral is again taken as the limit of a sum of products as the size of the volume element tends to zero. One obvious difference though is that the element of volume is a scalar (how could you define a direction with an infinitesimal volume element?). The possibilities are:

- $\int_V U(\mathbf{r})dV$ — scalar field; scalar integral.
- $\int_V \mathbf{a}dV$ — vector field; vector integral.

You have covered these (more or less) in your first year course, so not much more to say here. The next section considers these again in the context of a change of coordinates.

4.6 Changing variables: curvilinear coordinates

Up to now we have been concerned with Cartesian coordinates x, y, z with coordinate axes $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$. When performing a line integral in Cartesian coordinates, you write

$$\mathbf{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} \qquad \text{and} \qquad d\mathbf{r} = dx\hat{\boldsymbol{\imath}} + dy\hat{\boldsymbol{\jmath}} + dz\hat{\boldsymbol{k}}$$

and can be sure that length scales are properly handled because – as we saw in Lecture 3 –

$$|d\mathbf{r}| = ds = \sqrt{dx^2 + dy^2 + dz^2} \ .$$

The reason for using the basis $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$ rather than any other orthonormal basis set is that $\hat{\boldsymbol{\imath}}$ represents a direction in which x is increasing while the other two coordinates remain constant (and likewise for $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ with y and z respectively), simplifying the representation and resulting mathematics.

Often the symmetry of the problem strongly hints at using another coordinate system:

- likely to be plane, cylindrical, or spherical polars,
- but can be something more exotic

The general name for any different "u, v, w" coordinate system is a **curvilinear coordinate system**. We will see that the idea hinted at above – of defining a basis set by considering directions in which only one coordinate is (instantaneously) increasing – provides the approriate generalisation.

We begin by discussing common special cases: cylindrical polars and spherical polars, and conclude with a more general formulation.

4.6.1 Cylindrical polar coordinates

As shown in figure 4.4 a point in space P having cartesian coordinates x, y, z can be expressed in terms of cylindrical polar coordinates, r, ϕ, z as follows:

$$\begin{aligned} \mathbf{r} &= x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} \\ &= r\cos\phi\hat{\boldsymbol{\imath}} + r\sin\phi\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} \end{aligned}$$

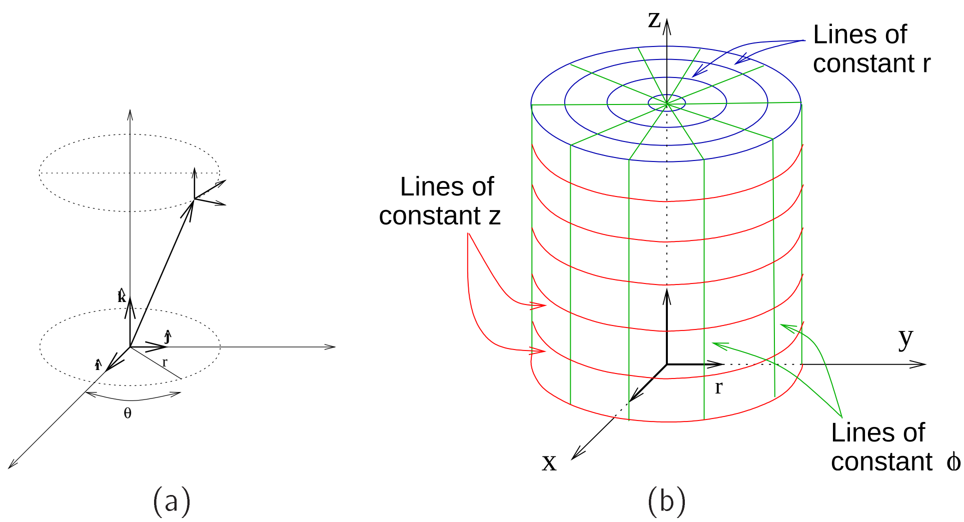


Figure 4.4: Cylindrical polars: (a) coordinate definition; (b) “iso” lines in r, ϕ and z.

Note that, by definition, $\frac{\partial \mathbf{r}}{\partial r}$ represents a direction in which (instantaneously) r is changing while the other two coordinates stay constant. That is, it is tangent to lines of constant ϕ and z. Likewise for $\frac{\partial \mathbf{r}}{\partial \phi}$ and $\frac{\partial \mathbf{r}}{\partial z}$, Thus the vectors:

$$
\begin{aligned}
\mathbf{e}_r &= \frac{\partial \mathbf{r}}{\partial r} = \cos\phi \hat{\boldsymbol{\imath}} + \sin\phi \hat{\boldsymbol{\jmath}} \\
\mathbf{e}_\phi &= \frac{\partial \mathbf{r}}{\partial \phi} = -r\sin\phi \hat{\boldsymbol{\imath}} + r\cos\phi \hat{\boldsymbol{\jmath}} \\
\mathbf{e}_z &= \frac{\partial \mathbf{r}}{\partial z} = \hat{\boldsymbol{k}}
\end{aligned}
$$

Aside on notation: some texts use the notation $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\phi}}, \ldots$ to represent the unit vectors that form the local basis set. Though I prefer the notation used here, where the basis vectors are written as $\hat{\mathbf{e}}$ with appropriate subscripts (as used in Riley *et al*), you should be aware of, and comfortable with, either possibility.

form a basis set in which we may describe infinitessimal vector displacements in the position of P, $d\mathbf{r}$. It is more usual, however, first to normalise the vectors to obtain their corresponding unit vectors, $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\phi$, $\hat{\mathbf{e}}_z$. Following the usual rules of calculus we may write:

$$
\begin{aligned}
d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial r}dr + \frac{\partial \mathbf{r}}{\partial \phi}d\phi + \frac{\partial \mathbf{r}}{\partial z}dz \\
&= dr\mathbf{e}_r + d\phi\mathbf{e}_\phi + dz\mathbf{e}_z \\
&= dr\hat{\mathbf{e}}_r + rd\phi\hat{\mathbf{e}}_\phi + dz\hat{\mathbf{e}}_z
\end{aligned}
$$

Now here is the important thing to note. In cartesian coordinates, a small change in (eg) x while keeping y and z constant would result in a displacement of

$$ds = |d\mathbf{r}| = \sqrt{d\mathbf{r}.d\mathbf{r}} = \sqrt{dx^2 + 0 + 0} = dx$$

But in cylindrical polars, a small change in ϕ of $d\phi$ while keeping r and z constant results in a displacement of

$$ds = |d\mathbf{r}| = \sqrt{r^2(d\phi)^2} = r d\phi$$

Thus the size of the (infinitessimal) displacement is dependent on the value of r. Factors such as this r are known as **scale factors** or **metric coefficients**, and we must be careful to take them into account when, eg, performing line, surface or volume integrals, as you will below. For cylindrical polars the metric coefficients are clearly $1, r$ and 1.

Example: line integral in cylindrical coordinates

Q Evaluate $\oint_C \mathbf{a} \cdot d\mathbf{l}$, where $\mathbf{a} = x^3\hat{\boldsymbol{\jmath}} - y^3\hat{\boldsymbol{\imath}} + x^2y\hat{\boldsymbol{k}}$ and C is the circle of radius r in the $z = 0$ plane, centred on the origin.

A Consider figure 4.5. In this case our cylindrical coordinates effectively reduce to plane polars since the path of integration is a circle in the $z = 0$ plane, but let's persist with the full set of coordinates anyway; the $\hat{\boldsymbol{k}}$ component of $\mathbf{a}$ will play no role (it is normal to the path of integration and therefore cancels as seen below).

On the circle of interest

$$\mathbf{a} = r^3(-\sin^3\phi\hat{\boldsymbol{\imath}} + \cos^3\phi\hat{\boldsymbol{\jmath}} + \cos^2\phi\sin\phi\hat{\boldsymbol{k}})$$

and (since dz = dr = 0 on the path)

$$\begin{aligned} d\mathbf{r} &= r\ d\phi\ \hat{\mathbf{e}}_\phi \\ &= r d\phi(-\sin\phi\hat{\boldsymbol{\imath}} + \cos\phi\hat{\boldsymbol{\jmath}}) \end{aligned}$$

so that

$$\oint_C \mathbf{a} \cdot d\mathbf{r} = \int_0^{2\pi} r^4(\sin^4\phi + \cos^4\phi)d\phi = \frac{3\pi}{2}r^4$$

since

$$\int_0^{2\pi} \sin^4\phi d\phi = \int_0^{2\pi} \cos^4\phi d\phi = \frac{3\pi}{4}$$

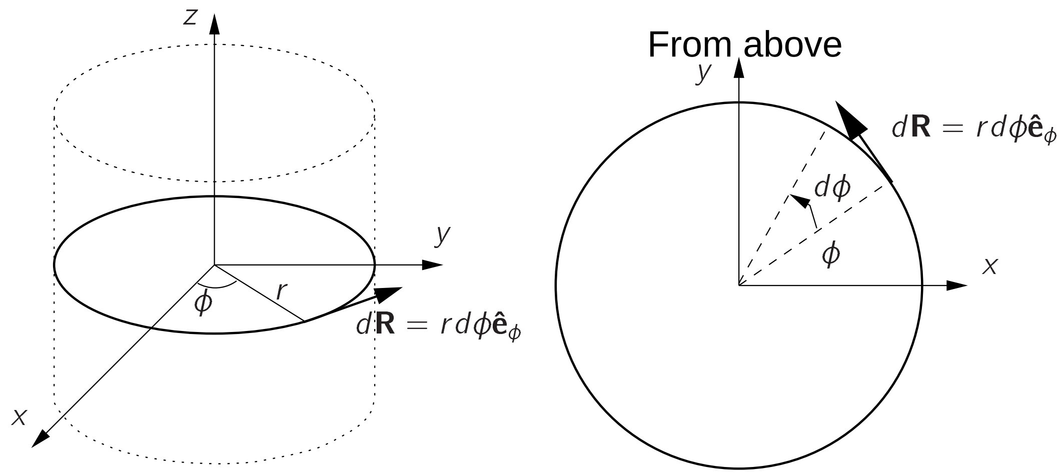


Figure 4.5: Line integral example in cylindrical coordinates

Volume integrals in cylindrical polars

In Cartesian coordinates a volume element is given by (see figure 4.6a):

$$dV = dxdydz$$

Recall that the volume of a parallelopiped is given by the scalar triple product of the vectors which define it (see section 2.1.2). Thus the formula above can be derived (even though it is "obvious") as:

$$dV = dx\hat{\boldsymbol{\imath}}.(dy\hat{\boldsymbol{\jmath}} \times dz\hat{\boldsymbol{k}}) = dxdydz$$

since the basis set is orthonormal.

In cylindrical polars a volume element is given by (see figure 4.6b):

$$dV = dr\hat{\mathbf{e}}_r.(rd\phi\hat{\mathbf{e}}_\phi \times dz\hat{\mathbf{e}}_z) = rd\phi drdz$$

Note also that this volume, because it is a scalar triple product, can be written as a determinant:

$$dV = \begin{vmatrix} \hat{\mathbf{e}}_r dr \\ \hat{\mathbf{e}}_\phi rd\phi \\ \hat{\mathbf{e}}_z dz \end{vmatrix} = \begin{vmatrix} \mathbf{e}_r dr \\ \mathbf{e}_\phi d\phi \\ \mathbf{e}_z dz \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} drd\phi dz$$

where the equality on the right-hand side follows from the definitions of $\hat{\mathbf{e}}_r = \frac{\partial \mathbf{r}}{\partial r} = \frac{\partial x}{\partial r}\hat{\boldsymbol{\imath}} + \frac{\partial y}{\partial r}\hat{\boldsymbol{\jmath}} + \frac{\partial z}{\partial r}\hat{\boldsymbol{k}}$, etc. This is the explanation for the "magical" appearance of the determinant in change-of-variables integration that you encountered in your first year maths!

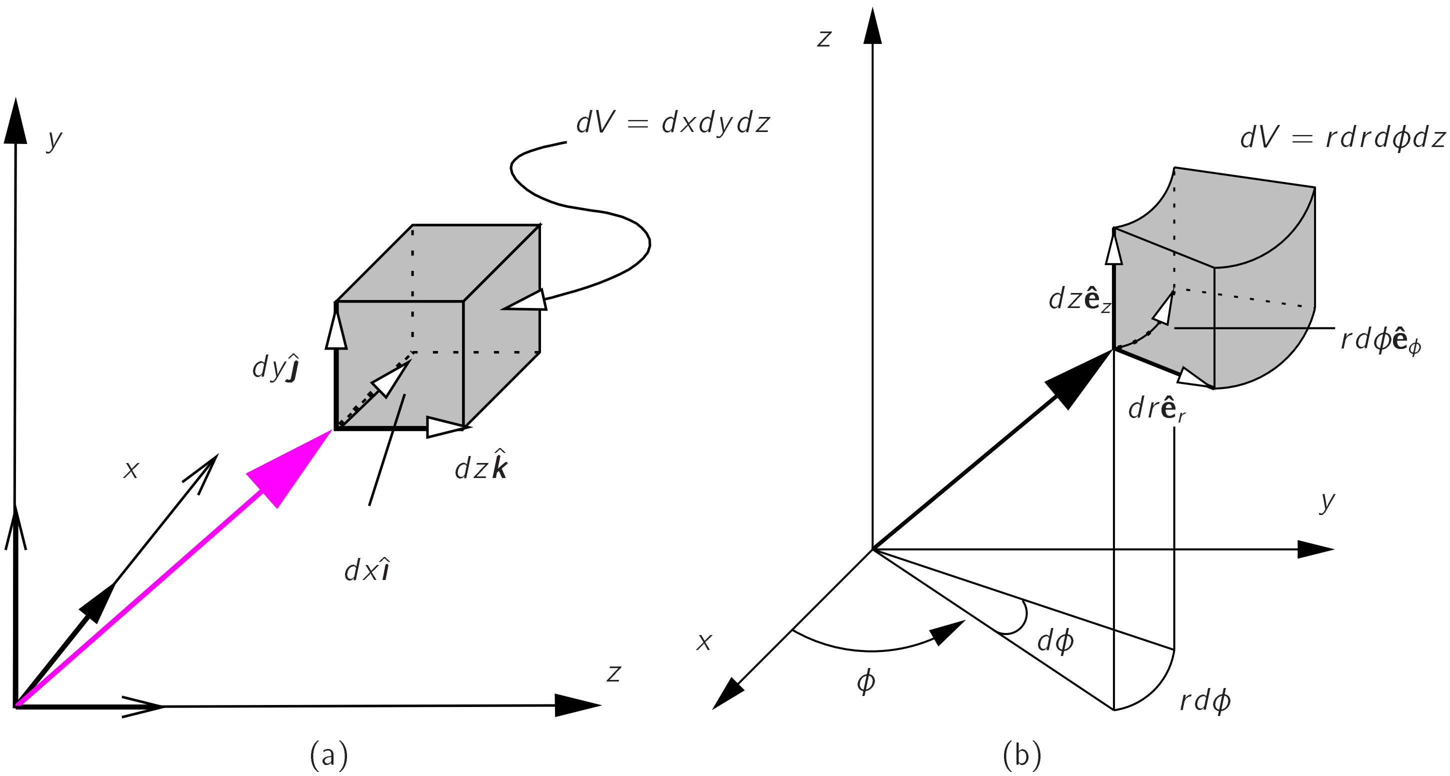


Figure 4.6: Volume elements dV in (a) Cartesian coordinates; (b) Cylindrical polar coordinates

Surface integrals in cylindrical polars

Recall from section 4.4 that for a surface element with normal along $\hat{\imath}$ we have:

$$d\mathbf{S} = dydz\hat{\imath}$$

More explictly this comes from finding normal to the plane that is tangent to the surface of constant x and from finding the area of an infinitessimal area element on the plane. In this case the plane is spanned by the vectors $\hat{\jmath}$ and $\hat{k}$ and the area of the element given by (see section 1.3):

$$dS = \left| dy\hat{\jmath} \times dz\hat{k} \right|$$

Thus

$$d\mathbf{S} = dy\hat{\jmath} \times dz\hat{k} = \hat{\imath} dS = dydz\hat{\imath}$$

In cylindrical polars, surface area elements (see figure 4.7) are given by:

$$\begin{aligned} d\mathbf{S} &= dr\hat{\mathbf{e}}_r \times rd\phi\hat{\mathbf{e}}_\phi = rdrd\phi\hat{\mathbf{e}}_z && \text{(for surfaces of constant } z\text{)} \\ d\mathbf{S} &= rd\phi\hat{\mathbf{e}}_\phi \times dz\hat{\mathbf{e}}_z = rd\phi dz\hat{\mathbf{e}}_r && \text{(for surfaces of constant } r\text{)} \end{aligned}$$

Similarly we can find $d\mathbf{S}$ for surfaces of constant ϕ, though since these aren't as common this is left as a (relatively easy) exercise.

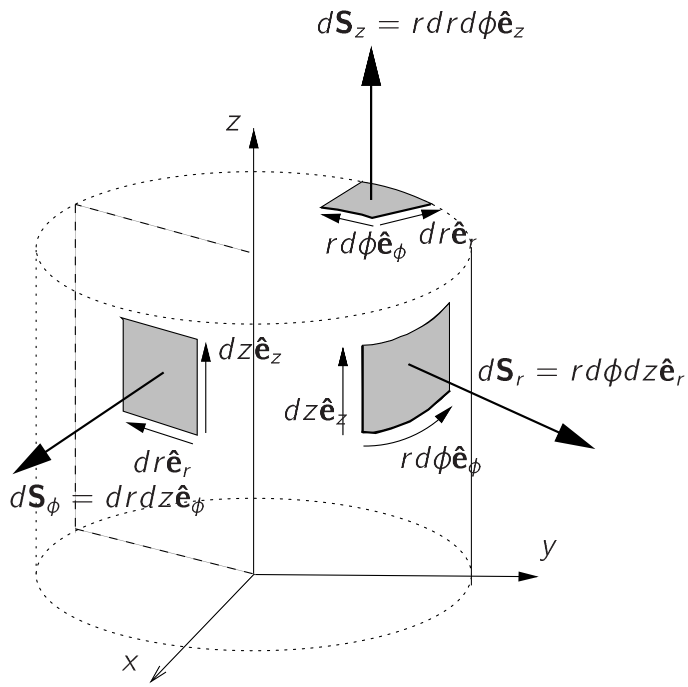


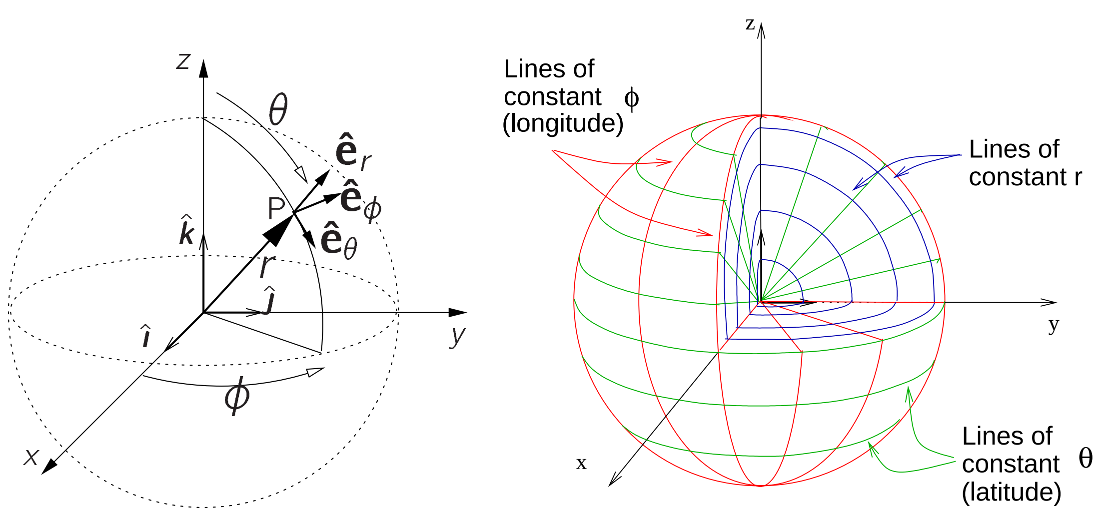
Figure 4.7: Surface elements in cylindrical polar coordinates

4.6.2 Spherical polars

Much of the development for spherical polars is similar to that for cylindrical polars. As shown in figure 4.6.2 a point in space P having cartesian coordinates x, y, z can be expressed in terms of spherical polar coordinates, r, θ, ϕ as follows:

$$\begin{aligned}\mathbf{r} &= x\hat{\imath} + y\hat{\jmath} + z\hat{\boldsymbol{k}} \\ &= r\sin\theta\cos\phi\hat{\imath} + r\sin\theta\sin\phi\hat{\jmath} + r\cos\theta\hat{\boldsymbol{k}}\end{aligned}$$

The basis set in spherical polars is obtained in an analogous fashion: we find unit



vectors which are in the direction of increase of each coordinate:

$$\begin{aligned}
\mathbf{e}_r = \frac{\partial \mathbf{r}}{\partial r} &= \sin\theta\cos\phi\hat{\boldsymbol{\imath}} + \sin\theta\sin\phi\hat{\boldsymbol{\jmath}} + \cos\theta\hat{\boldsymbol{k}} &= \hat{\mathbf{e}}_r \\
\mathbf{e}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} &= r\cos\theta\cos\phi\hat{\boldsymbol{\imath}} + r\cos\theta\sin\phi\hat{\boldsymbol{\jmath}} - r\sin\theta\hat{\boldsymbol{k}} &= r\hat{\mathbf{e}}_\theta \\
\mathbf{e}_\phi = \frac{\partial \mathbf{r}}{\partial \phi} &= -r\sin\theta\sin\phi\hat{\boldsymbol{\imath}} + r\sin\theta\cos\phi\hat{\boldsymbol{\jmath}} &= r\sin\theta\hat{\mathbf{e}}_\phi
\end{aligned}$$

As with cylindrical polars, it is easily verified that the vectors $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi$ form an orthonormal basis.

A small displacement $d\mathbf{r}$ is given by:

$$\begin{aligned}
d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial r}dr + \frac{\partial \mathbf{r}}{\partial \theta}d\theta + \frac{\partial \mathbf{r}}{\partial \phi}d\phi \\
&= dr\mathbf{e}_r + d\theta\mathbf{e}_\theta + d\phi\mathbf{e}_\phi \\
&= dr\hat{\mathbf{e}}_r + rd\theta\hat{\mathbf{e}}_\theta + r\sin\theta d\phi\hat{\mathbf{e}}_\phi
\end{aligned}$$

Thus the metric coefficients are $1, r, r\sin\theta$.

Volume integrals in spherical polars

In spherical polars a volume element is given by (see figure 4.8):

$$dV = dr\hat{\mathbf{e}}_r.(rd\theta\hat{\mathbf{e}}_\theta \times r\sin\theta d\phi\hat{\mathbf{e}}_\phi) = r^2\sin\theta dr d\theta d\phi$$

Note again that this volume could be written as a determinant, but this is left as an exercise.

Surface integrals in spherical polars

The most (the only?) useful surface elements in spherical polars are those tangent to surfaces of constant r (see figure 4.9). The surface direction (unnormalised) is given by $\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_\phi = \hat{\mathbf{e}}_r$ and the area of an infinitessimal surface element is given by $|rd\theta\hat{\mathbf{e}}_\theta \times r\sin\theta d\phi\hat{\mathbf{e}}_\phi| = r^2\sin\theta d\theta d\phi$.

Thus a surface element $d\mathbf{S}$ in spherical polars is given by

$$d\mathbf{S} = rd\theta\hat{\mathbf{e}}_\theta \times r\sin\theta d\phi\hat{\mathbf{e}}_\phi = r^2\sin\theta\hat{\mathbf{e}}_r$$

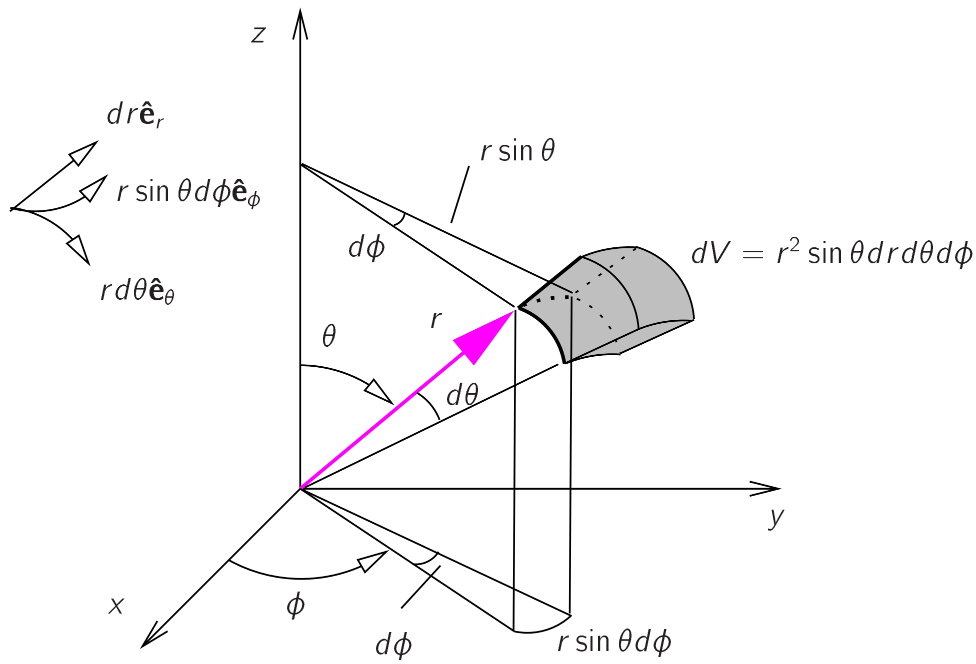


Figure 4.8: Volume element dV in spherical polar coordinates

♣ **Example: surface integral in spherical polars**

Q Evaluate $\int_S \mathbf{a}\cdot d\mathbf{S}$, where $\mathbf{a} = z^3\hat{\boldsymbol{k}}$ and S is the sphere of radius A centred on the origin.

A On the surface of the sphere:

$$\mathbf{a} = A^3 cos^3\theta\hat{\boldsymbol{k}} \qquad d\mathbf{S} = A^2\sin\theta\, d\theta\, d\phi\hat{\mathbf{e}}_r$$

Hence

$$\begin{aligned}
\int_S \mathbf{a}\cdot d\mathbf{S} &= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} A^3 cos^3\theta\ A^2\sin\theta\ [\hat{\mathbf{e}}_r\cdot\hat{\boldsymbol{k}}]\ d\theta d\phi \\
&= A^5\int_0^{2\pi} d\phi\ \int_0^{\pi} cos^3\theta\sin\theta[\cos\theta]\ d\theta \\
&= 2\pi A^5\frac{1}{5}\left[-\cos^5\theta\right]_0^{\pi} \\
&= \frac{4\pi A^5}{5}
\end{aligned}$$

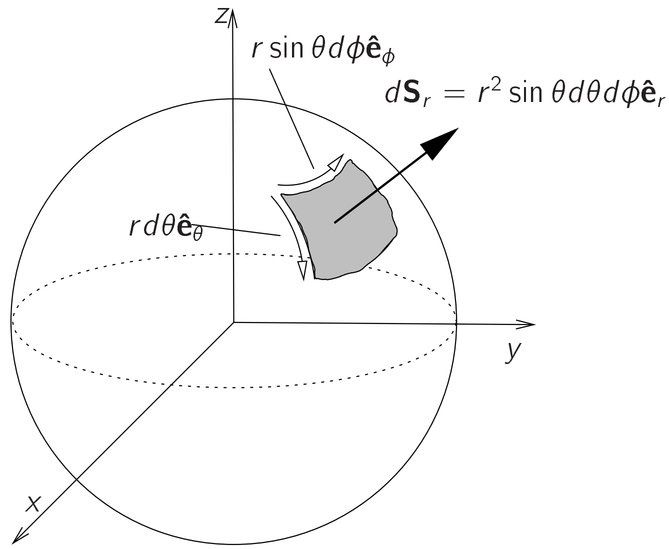


Figure 4.9: Surface element $d\mathbf{S}$ in spherical polar coordinates

4.6.3 General curvilinear coordinates

Cylindrical and spherical polar coordinates are two (useful) examples of general curvilinear coordinates. In general a point P with Cartesian coordinates x, y, z can be expressed in the terms of the curvilinear coordinates u, v, w where

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w)$$

Thus

$$\mathbf{r} = x(u, v, w)\hat{\imath} + y(u, v, w)\hat{\jmath} + z(u, v, w)\hat{\boldsymbol{k}}$$

and

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\hat{\imath} + \frac{\partial y}{\partial u}\hat{\jmath} + \frac{\partial z}{\partial u}\hat{\boldsymbol{k}}$$

and similarly for partials with respect to v and w, so

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}du + \frac{\partial \mathbf{r}}{\partial v}dv + \frac{\partial \mathbf{r}}{\partial w}dw$$

We now define the local coordinate system as before by considering the directions in which each coordinate "unilaterally" (and instantaneously) increases:

$$\begin{aligned}
\mathbf{e}_u &= \frac{\partial \mathbf{r}}{\partial u} = \left|\frac{\partial \mathbf{r}}{\partial u}\right| \hat{\mathbf{e}}_u = h_u \hat{\mathbf{e}}_u \\
\mathbf{e}_v &= \frac{\partial \mathbf{r}}{\partial v} = \left|\frac{\partial \mathbf{r}}{\partial v}\right| \hat{\mathbf{e}}_v = h_v \hat{\mathbf{e}}_v \\
\mathbf{e}_w &= \frac{\partial \mathbf{r}}{\partial w} = \left|\frac{\partial \mathbf{r}}{\partial w}\right| \hat{\mathbf{e}}_w = h_w \hat{\mathbf{e}}_w
\end{aligned}$$

The metric coefficients are therefore $h_u = |\frac{\partial \mathbf{r}}{\partial u}|$, $h_v = |\frac{\partial \mathbf{r}}{\partial v}|$ and $h_w = |\frac{\partial \mathbf{r}}{\partial w}|$.
A volume element is in general given by

$$dV = h_u du \hat{\mathbf{e}}_u . (h_v dv \hat{\mathbf{e}}_v \times h_w dw \hat{\mathbf{e}}_w)$$

and simplifies *if the coordinate system is orthonormal* (since $\hat{\mathbf{e}}_u.(\hat{\mathbf{e}}_v \times \hat{\mathbf{e}}_w) = 1$) to

$$dV = h_u h_v h_w du dv dw$$

A surface element (normal to constant w, say) is in general

$$d\mathbf{S} = h_u du \hat{\mathbf{e}}_u \times h_v dv \hat{\mathbf{e}}_v$$

and simplifies *if the coordinate system is orthogonal* to

$$d\mathbf{S} = h_u h_v du dv \hat{\mathbf{e}}_w$$

4.6.4 Summary

To summarise:

General curvilinear coordinates

$$\begin{aligned}
x &= x(u,v,w), \qquad y = y(u,v,w), \qquad z = z(u,v,w) \\
\mathbf{r} &= x(u,v,w)\hat{\boldsymbol{\imath}} + y(u,v,w)\hat{\boldsymbol{\jmath}} + z(u,v,w)\hat{\boldsymbol{k}} \\
h_u &= \left|\frac{\partial \mathbf{r}}{\partial u}\right|, \qquad h_v = \left|\frac{\partial \mathbf{r}}{\partial v}\right|, \qquad h_w = \left|\frac{\partial \mathbf{r}}{\partial w}\right| \\
\hat{\mathbf{u}} &= \hat{\mathbf{e}}_u = \frac{1}{h_u}\frac{\partial \mathbf{r}}{\partial u}, \qquad \hat{\mathbf{v}} = \hat{\mathbf{e}}_v = \frac{1}{h_v}\frac{\partial \mathbf{r}}{\partial v}, \qquad \hat{\mathbf{w}} = \hat{\mathbf{e}}_w = \frac{1}{h_w}\frac{\partial \mathbf{r}}{\partial w} \\
d\mathbf{r} &= h_u du \hat{\mathbf{u}} + h_v dv \hat{\mathbf{v}} + h_w dw \hat{\mathbf{w}} \\
dV &= h_u h_v h_w du dv dw \; \hat{\mathbf{u}}.(\hat{\mathbf{v}} \times \hat{\mathbf{w}}) \\
d\mathbf{S} &= h_u h_v du dv \; \hat{\mathbf{u}} \times \hat{\mathbf{v}} \quad \text{(for surface element tangent to constant } w\text{)}
\end{aligned}$$

Plane polar coordinates

$$\begin{aligned}
x &= r\cos\theta, \qquad y = r\sin\theta \\
\mathbf{r} &= r\cos\theta\hat{\boldsymbol{\imath}} + r\sin\theta\hat{\boldsymbol{\jmath}} \\
h_r &= 1, \qquad h_\theta = r \\
\hat{\mathbf{e}}_r &= \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}, \qquad \hat{\mathbf{e}}_\theta = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}} \\
d\mathbf{r} &= dr\hat{\mathbf{e}}_r + r d\theta \hat{\mathbf{e}}_\theta \\
d\mathbf{S} &= r dr d\theta \hat{\boldsymbol{k}}
\end{aligned}$$

Cylindrical polar coordinates

$$
\begin{aligned}
x &= r\cos\phi, \qquad y = r\sin\phi, \qquad z = z \\
\mathbf{r} &= r\cos\phi\hat{\boldsymbol{\imath}} + r\sin\phi\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} \\
h_r &= 1, \qquad h_\phi = r, \qquad h_z = 1 \\
\hat{\mathbf{e}}_r &= \cos\phi\hat{\boldsymbol{\imath}} + \sin\phi\hat{\boldsymbol{\jmath}}, \qquad \hat{\mathbf{e}}_\phi = -\sin\phi\hat{\boldsymbol{\imath}} + \cos\phi\hat{\boldsymbol{\jmath}}, \qquad \hat{\mathbf{e}}_z = \hat{\boldsymbol{k}} \\
d\mathbf{r} &= dr\hat{\mathbf{e}}_r + rd\phi\hat{\mathbf{e}}_\phi + dz\hat{\mathbf{e}}_z \\
d\mathbf{S} &= rdrd\phi\hat{\boldsymbol{k}} \quad \text{(on the flat ends)} \\
d\mathbf{S} &= rd\phi dz\hat{\mathbf{e}}_r \quad \text{(on the curved sides)} \\
dV &= rdrd\phi dz
\end{aligned}
$$

Spherical polar coordinates

$$
\begin{aligned}
x &= r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta \\
\mathbf{r} &= r\sin\theta\cos\phi\hat{\boldsymbol{\imath}} + r\sin\theta\sin\phi\hat{\boldsymbol{\jmath}} + r\cos\theta\hat{\boldsymbol{k}} \\
h_r &= 1, \qquad h_\theta = r, \qquad h_\phi = r\sin\theta \\
\hat{\mathbf{e}}_r &= \sin\theta\cos\phi\hat{\boldsymbol{\imath}} + \sin\theta\sin\phi\hat{\boldsymbol{\jmath}} + \cos\theta\hat{\boldsymbol{k}} \\
\hat{\mathbf{e}}_\theta &= \cos\theta\cos\phi\hat{\boldsymbol{\imath}} + \cos\theta\sin\phi\hat{\boldsymbol{\jmath}} + \sin\theta\hat{\boldsymbol{k}} \\
\hat{\mathbf{e}}_\phi &= -\sin\phi\hat{\boldsymbol{\imath}} + \cos\phi\hat{\boldsymbol{\jmath}} \\
d\mathbf{r} &= dr\hat{\mathbf{e}}_r + rd\theta\hat{\mathbf{e}}_\theta + r\sin\theta d\phi\hat{\mathbf{e}}_\phi \\
d\mathbf{S} &= r^2\sin\theta drd\theta d\phi\hat{\mathbf{e}}_r \quad \text{(on a spherical surface)} \\
dV &= r^2\sin\theta drd\theta d\phi
\end{aligned}
$$

Lecture 5

Vector Operators: Grad, Div and Curl

In the first lecture of the second part of this course we move more to consider properties of fields. We introduce three field operators which reveal interesting collective field properties, viz.

- the **gradient** of a scalar field,
- the **divergence** of a vector field, and
- the **curl** of a vector field.

There are two points to get over about each:

- The mechanics of taking the grad, div or curl, for which you will need to brush up your multivariate calculus.
- The underlying physical meaning — that is, why they are worth bothering about.

In Lecture 6 we will look at combining these vector operators.

5.1 The gradient of a scalar field

Recall the discussion of temperature distribution throughout a room in the overview, where we wondered how a scalar would vary as we moved off in an arbitrary direction. Here we find out how.

If $U(x, y, z)$ is a scalar field, ie a scalar function of position $\mathbf{r} = [x, y, z]$ in 3 dimensions, then its **gradient** at any point is defined in Cartesian co-ordinates by

$$\text{grad}U = \frac{\partial U}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial U}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial U}{\partial z}\hat{\boldsymbol{k}} .$$

It is usual to define the **vector operator** which is called "del" or "nabla"

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\imath}}\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z} .$$

Then

$$\text{grad}U \equiv \nabla U \; .$$

Note immediately that ∇U is a vector field!

Without thinking too carefully about it, we can see that the gradient of a scalar field tends to point in the direction of greatest change of the field. Later we will be more precise.

♣ Worked examples of gradient evaluation

1. $U = x^2$

$$\Rightarrow \nabla U = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}}\right) x^2 = 2x\hat{\imath} \; .$$

2. $U = r^2$

$$\begin{aligned} r^2 &= x^2 + y^2 + z^2 \\ \Rightarrow \nabla U &= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}}\right)(x^2+y^2+z^2) \\ &= 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{\boldsymbol{k}} = 2\,\mathbf{r} \; . \end{aligned}$$

3. $U = \mathbf{c}\cdot\mathbf{r}$, where $\mathbf{c}$ is constant.

$$\Rightarrow \nabla U = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}\right)(c_1 x + c_2 y + c_3 z) = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{\boldsymbol{k}} = \mathbf{c} \; .$$

4. $U = f(r)$, where $r = \sqrt{(x^2+y^2+z^2)}$

U is a function of r alone so df/dr exists. As $U = f(x,y,z)$ also,

$$\frac{\partial f}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x} \qquad \frac{\partial f}{\partial y} = \frac{df}{dr}\frac{\partial r}{\partial y} \qquad \frac{\partial f}{\partial z} = \frac{df}{dr}\frac{\partial r}{\partial z} \; .$$

$$\Rightarrow \nabla U = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}} = \frac{df}{dr}\left(\frac{\partial r}{\partial x}\hat{\imath} + \frac{\partial r}{\partial y}\hat{\jmath} + \frac{\partial r}{\partial z}\hat{\boldsymbol{k}}\right)$$

But $r = \sqrt{x^2+y^2+z^2}$, so $\partial r/\partial x = x/r$ and similarly for y, z.

$$\Rightarrow \nabla U = \frac{df}{dr}\left(\frac{x\hat{\imath} + y\hat{\jmath} + z\hat{\boldsymbol{k}}}{r}\right) = \frac{df}{dr}\left(\frac{\mathbf{r}}{r}\right) \; .$$

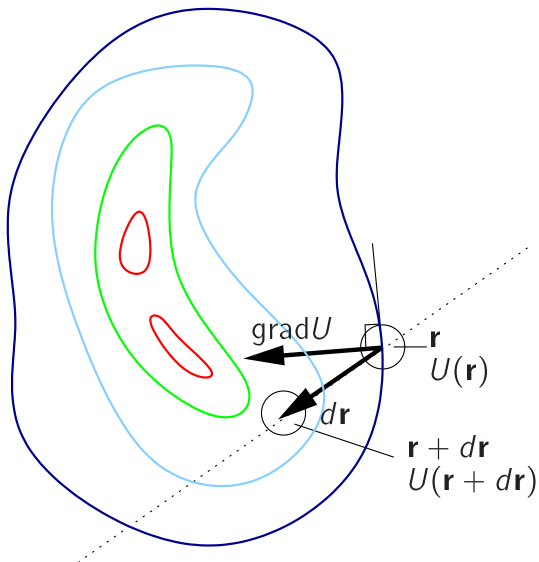


Figure 5.1: The directional derivative

5.2 The significance of grad

If our current position is $\mathbf{r}$ in some scalar field U (Fig. 5.1), and we move an infinitesimal distance $d\mathbf{r}$, we know that the change in U is

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \ .$$

But we know that $d\mathbf{r} = (\hat{\imath}dx + \hat{\jmath}dy + \hat{\mathbf{k}}dz)$ and $\nabla U = (\hat{\imath}\partial U/\partial x + \hat{\jmath}\partial U/\partial y + \hat{\mathbf{k}}\partial U/\partial z)$, so that the change in U is also given by the scalar product

$$dU = \nabla U \cdot d\mathbf{r} \ .$$

Now divide both sides by ds

$$\frac{dU}{ds} = \nabla U \cdot \frac{d\mathbf{r}}{ds} \ .$$

But remember that $|d\mathbf{r}| = ds$, so $d\mathbf{r}/ds$ is a unit vector in the direction of $d\mathbf{r}$.

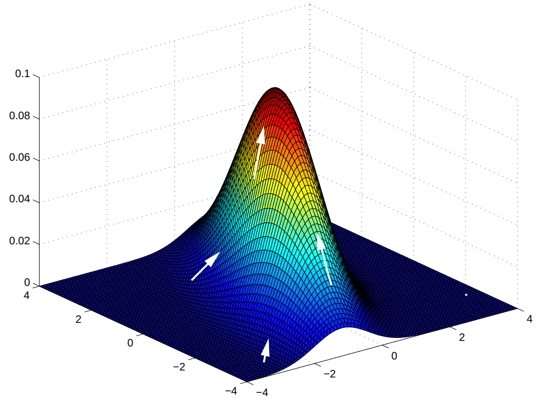
This result can be paraphrased as:

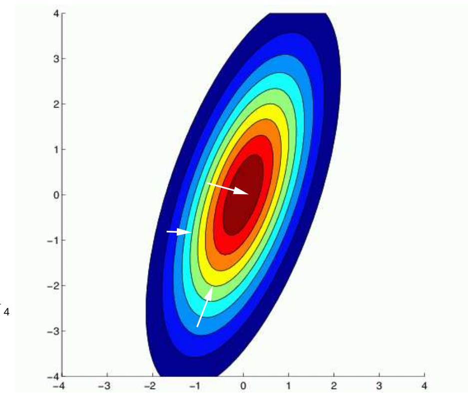
- grad$U$ has the property that the rate of change of U wrt distance in a particular direction ($\hat{\mathbf{d}}$) is the projection of grad$U$ onto that direction (or the component of grad$U$ in that direction).

The quantity dU/ds is called a **directional derivative**, but note that in general it has a different value for each direction, and so has no meaning until you specify the direction.

We could also say that

- At any point P, grad$U$ points in the direction of greatest change of U at P, and has magnitude equal to the rate of change of U wrt distance in that direction.





Another nice property emerges if we think of a surface of constant U – that is the locus (x, y, z) for

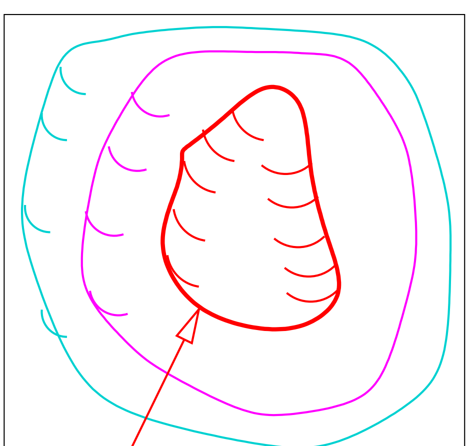
$$U(x, y, z) = \text{constant} .$$

If we move a tiny amount within that iso-U surface, there is no change in U, so $dU/ds = 0$. So for any $d\mathbf{r}/ds$ in the surface

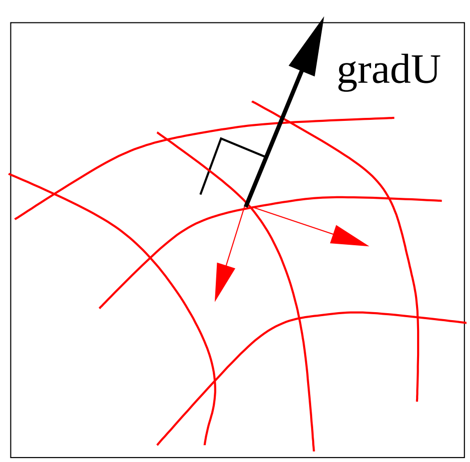
$$\nabla U \cdot \frac{d\mathbf{r}}{ds} = 0 .$$

But $d\mathbf{r}/ds$ is a tangent to the surface, so this result shows that

- grad$U$ is everywhere NORMAL to a surface of constant U.



Surface of constant U
These are called Level Surfaces



Surface of constant U

5.3 The divergence of a vector field

The divergence computes a scalar quantity from a vector field by differentiation.

If $\mathbf{a}(x, y, z)$ is a vector function of position in 3 dimensions, that is $\mathbf{a} = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$, then its divergence at any point is defined in Cartesian co-ordinates by

$$\text{div}\mathbf{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

We can write this in a simplified notation using a scalar product with the $\boldsymbol{\nabla}$ vector differential operator:

$$\text{div}\mathbf{a} = \left(\hat{\boldsymbol{\imath}}\,\frac{\partial}{\partial x} \;+\; \hat{\boldsymbol{\jmath}}\,\frac{\partial}{\partial y} \;+\; \hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}\right) \cdot \mathbf{a} = \boldsymbol{\nabla} \cdot \mathbf{a}$$

Notice that the divergence of a vector field is a scalar field.

♣ Examples of divergence evaluation

	$\mathbf{a}$	div$\mathbf{a}$
1)	$x\hat{\boldsymbol{\imath}}$	1
2)	$\mathbf{r}(= x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}})$	3
3)	$\mathbf{r}/r^3$	0
4)	$r\mathbf{c}$, for $\mathbf{c}$ constant	$(\mathbf{r}\cdot\mathbf{c})/r$

We work through example 3).

The x component of $\mathbf{r}/r^3$ is $x.(x^2 + y^2 + z^2)^{-3/2}$, and we need to find $\partial/\partial x$ of it.

$$\begin{aligned}\frac{\partial}{\partial x}x.(x^2 + y^2 + z^2)^{-3/2} &= 1.(x^2 + y^2 + z^2)^{-3/2} + x\frac{-3}{2}(x^2 + y^2 + z^2)^{-5/2}.2x \\ &= r^{-3}\left(1 - 3x^2r^{-2}\right) \;.\end{aligned}$$

The terms in y and z are similar, so that

$$\begin{aligned}\text{div}(\mathbf{r}/r^3) &= r^{-3}\left(3 - 3(x^2 + y^2 + z^2)r^{-2}\right) = r^{-3}\left(3 - 3\right) \\ &= 0\end{aligned}$$

5.4 The significance of div

Consider a typical vector field, water flow, and denote it by $\mathbf{a}(\mathbf{r})$. This vector has magnitude equal to the mass of water crossing a unit area perpendicular to the direction of $\mathbf{a}$ per unit time.

Now take an infinitesimal volume element dV and figure out the balance of the flow of $\mathbf{a}$ in and out of dV.

To be specific, consider the volume element $dV = dxdydz$ in Cartesian coordinates, and think first about the face of area $dxdz$ perpendicular to the y axis and facing outwards in the negative y direction. (That is, the one with surface area $d\mathbf{S} = -dxdz\hat{\boldsymbol{\jmath}}$.)

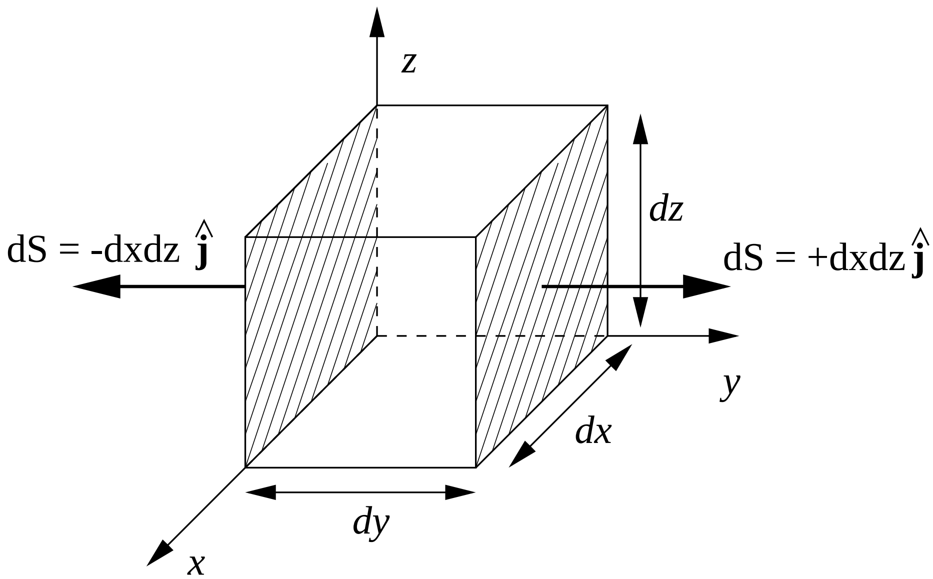


Figure 5.2: Elemental volume for calculating divergence.

The component of the vector **a** normal to this face is $\mathbf{a}\cdot\hat{\boldsymbol{\jmath}} = a_y$, and is pointing inwards, and so the its contribution to the OUTWARD flux from this surface is

$$\mathbf{a}\cdot d\mathbf{S} \quad = \quad -\, a_y(y)dzdx \;,$$

where $a_y(y)$ means that a_y is a function of y. (By the way, flux here denotes mass per unit time.)

A similar contribution, but of opposite sign, will arise from the opposite face, but we must remember that we have moved along y by an amount dy, so that this OUTWARD amount is

$$a_y(y+dy)dzdx = \left(a_y + \frac{\partial a_y}{\partial y}dy\right) dxdz$$

The total outward amount from these two faces is

$$\frac{\partial a_y}{\partial y}dydxdz = \frac{\partial a_y}{\partial y}dV$$

Summing the other faces gives a total outward flux of

$$\left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right) dV = \boldsymbol{\nabla}\cdot\mathbf{a} \;\; dV$$

So we see that

The divergence of a vector field represents the flux generation per unit volume at each point of the field. (Divergence because it is an efflux not an influx.)

Interestingly we also saw that the total efflux from the infinitesimal volume was equal to the flux integrated over the surface of the volume.

(NB: The above does not constitute a rigorous proof of the assertion because we have not proved that the quantity calculated is independent of the co-ordinate system used, but it will suffice for our purposes.)

5.5 The Laplacian: div(grad$U$) of a scalar field

Recall that grad$U$ of *any* scalar field U is a vector field. Recall also that we can compute the divergence of any vector field. So we can certainly compute div(grad$U$), even if we don't know what it means yet.

Here is where the ∇ operator starts to be really handy.

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}U) &= \left(\hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}\right)\cdot\left(\left(\hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}\right)U\right)\\
&= \left(\left(\hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}\right)\cdot\left(\hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}\right)\right)U\\
&= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)U\\
&= \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)
\end{aligned}$$

This last expression occurs frequently in engineering science (you will meet it next in solving Laplace's Equation in partial differential equations). For this reason, the operator ∇^2 is called the "Laplacian"

$$\nabla^2 U = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)U$$

Laplace's equation itself is

$$\nabla^2 U = 0$$

♣ Examples of $\nabla^2 U$ evaluation

	U	$\nabla^2 U$
1)	$r^2 (= x^2 + y^2 + z^2)$	6
2)	xy^2z^3	$2xz^3 + 6xy^2z$
3)	$1/r$	0

Let's prove example (3) (which is particularly significant – can you guess why?).

$$1/r = (x^2 + y^2 + z^2)^{-1/2}$$

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{-1/2} &= \frac{\partial}{\partial x} - x.(x^2 + y^2 + z^2)^{-3/2} \\
&= -(x^2 + y^2 + z^2)^{-3/2} + 3x.x.(x^2 + y^2 + z^2)^{-5/2} \\
&= (1/r^3)(-1 + 3x^2/r^2)
\end{aligned}$$

Adding up similar terms for y and z

$$\nabla^2 \frac{1}{r} = \frac{1}{r^3}\left(-3 + 3\frac{(x^2 + y^2 + x^2)}{r^2}\right) = 0$$

5.6 The curl of a vector field

So far we have seen the operator $\boldsymbol{\nabla}$ applied to a scalar field $\boldsymbol{\nabla} U$; and dotted with a vector field $\boldsymbol{\nabla} \cdot \mathbf{a}$.

We are now overwhelmed by an irrestible temptation to

- cross it with a vector field $\boldsymbol{\nabla} \times \mathbf{a}$

This gives the **curl of a vector field**

$$\boldsymbol{\nabla} \times \mathbf{a} \equiv \text{curl}(\mathbf{a})$$

We can follow the pseudo-determinant recipe for vector products, so that

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{a} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} \qquad \text{(remember it this way)} \\
&= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial y}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\hat{\boldsymbol{k}}
\end{aligned}$$

♣ Examples of curl evaluation

	$\mathbf{a}$	$\nabla \times \mathbf{a}$
1)	$-y\hat{\imath} + x\hat{\jmath}$	$2\hat{\mathbf{k}}$
2)	$x^2y^2\hat{\mathbf{k}}$	$2x^2y\hat{\imath} - 2xy^2\hat{\jmath}$

5.7 The signficance of curl

Perhaps the first example gives a clue. The field $\mathbf{a} = -y\hat{\imath} + x\hat{\jmath}$ is sketched in Figure 5.3(a). (It is the field you would calculate as the velocity field of an object rotating with $\boldsymbol{\omega} = [0, 0, 1]$.) This field has a curl of $2\hat{\mathbf{k}}$, which is in the r-h screw sense out of the page. You can also see that a field like this must give a finite value to the line integral around the complete loop $\oint_C \mathbf{a} \cdot d\mathbf{r}$.

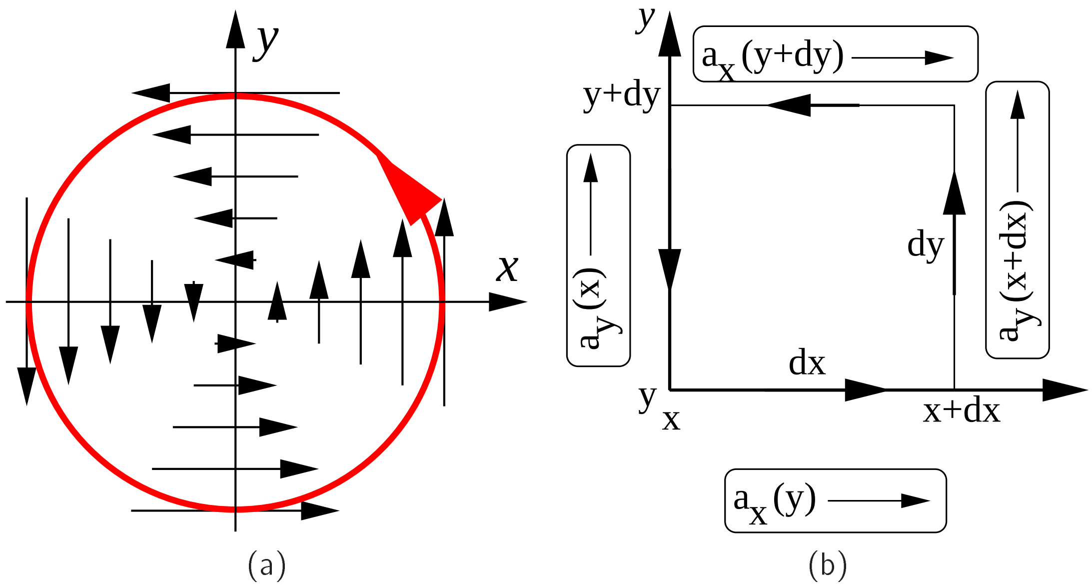


Figure 5.3: (a) A rough sketch of the vector field $-y\hat{\imath} + x\hat{\jmath}$. (b) An element in which to calculate curl.

In fact curl is closely related to the line integral around a loop.

> The **circulation** of a vector **a** round any closed curve C is defined to be $\oint_C \mathbf{a} \cdot d\mathbf{r}$
> and the **curl** of the vector field **a** represents the **vorticity**, or **circulation per unit area**, of the field.

Our proof uses the small rectangular element dx by dy shown in Figure 5.3(b). Consider the circulation round the perimeter of a rectangular element.

The fields in the x direction at the bottom and top are

$$a_x(y) \qquad \text{and} \qquad a_x(y+dy) = a_x(y) + \frac{\partial a_x}{\partial y} dy,$$

where $a_x(y)$ denotes a_x is a function of y, and the fields in the y direction at the left and right are

$$a_y(x) \qquad \text{and} \qquad a_y(x+dx) = a_y(x) + \frac{\partial a_y}{\partial x} dx$$

Starting at the bottom and working round in the anticlockwise sense, the four contributions to the circulation dC are therefore as follows, where the minus signs take account of the path being opposed to the field:

$$\begin{aligned} dC &= +[a_x(y)\ dx] + [a_y(x+dx)\ dy] - [a_x(y+dy)\ dx] - [a_y(x)\ dy] \\ &= +[a_x(y)\ dx] + \left[\left(a_y(x) + \frac{\partial a_y}{\partial x} dx\right) dy\right] - \left[\left(a_x(y) + \frac{\partial a_x}{\partial y} dy\right) dx\right] - [a_y(x)\ dy] \\ &= \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\ dx\ dy \\ &= (\nabla \times \mathbf{a}) \cdot d\mathbf{S} \end{aligned}$$

where $d\mathbf{S} = dxdy\hat{\boldsymbol{k}}$.

NB: Again, this is not a completely rigorous proof as we have not shown that the result is independent of the co-ordinate system used.

5.8 Some definitions involving div, curl and grad

- A vector field with zero divergence is said to be **solenoidal**.
- A vector field with zero curl is said to be **irrotational**.
- A scalar field with zero gradient is said to be, er, **constant**.