3: Convex Hulls and Conics & Quadrics

- Lecture 1: Euclidean, similarity, affine and projective transformations. Homogeneous coordinates and matrices. Coordinate frames. Perspective projection and its matrix representation.

- Lecture 2: Vanishing points. Horizons. Applications of projective transformations.

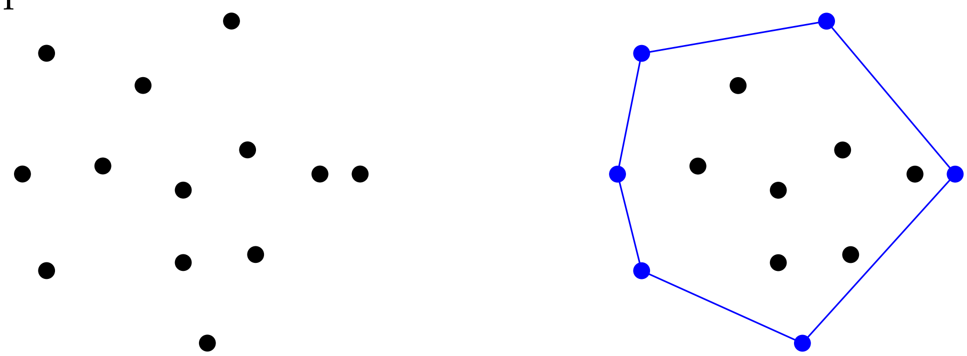
- **Lecture 3**: **Convexity of point-sets, convex hull and algorithms. Conics and quadrics, implicit and parametric forms, computation of intersections.**

- Lecture 4: Bezier curves, B-splines. Tensor-product surfaces.

Definition of convex hull 3.1

Informal definition (in 2D):

The convex hull of a set of points is the shape taken by a *rubber band* wrapped around the points.

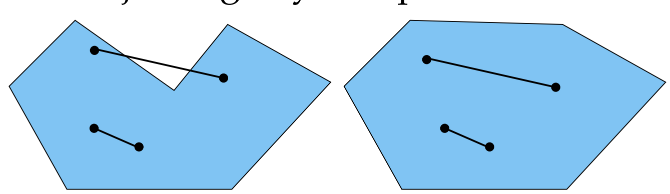


Formal definition (in 2D):

The convex hull is the smallest **convex** polygon that contains the points.

Convex sets of points 3.2

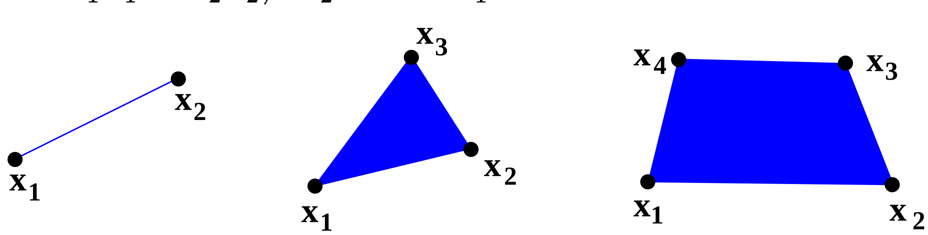
A set S is **convex** if the line joining any two points in S is contained in S.



Given points $\mathbf{x}_1,\ \mathbf{x}_2, \ldots\ \mathbf{x}_n$, the following generates a convex set:

$$\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i \quad \text{with} \quad \alpha_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i = 1$$

e.g. $n = 2, \quad x = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2, \quad \alpha_2 = 1 - \alpha_1$

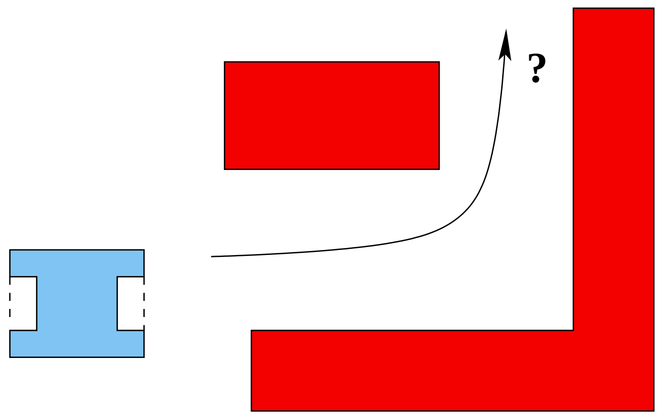


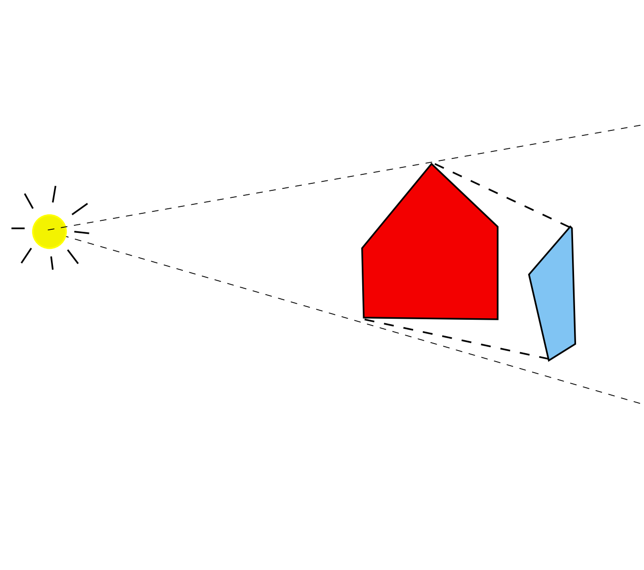
n = 2 n = 3 n = 4

Applications 3.3

2. Shadows

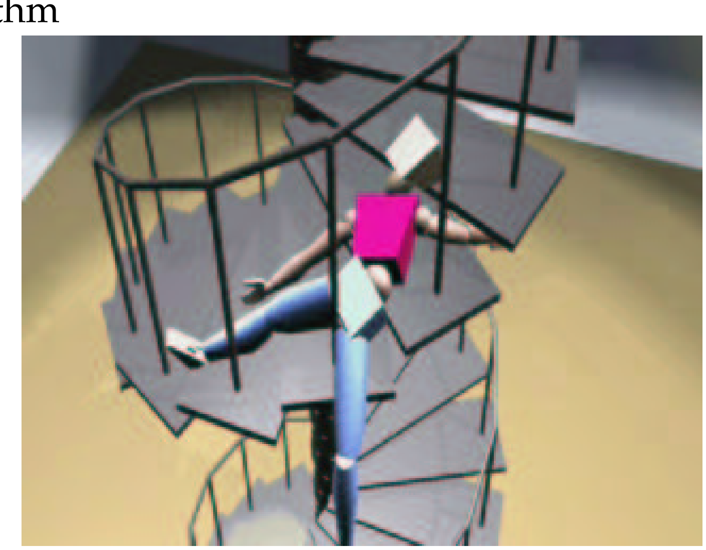
1. Robot path planning





Convex Hull Graphics Example 3.4

• Testing against the hull is quicker, for example "falling bodies video" uses the ' **quick hull** algorithm



Convex hull algorithms 3.5

Problem: Given a set of n points find their convex hull.

We will look at four algorithms and in each case examine their **time complexity**.

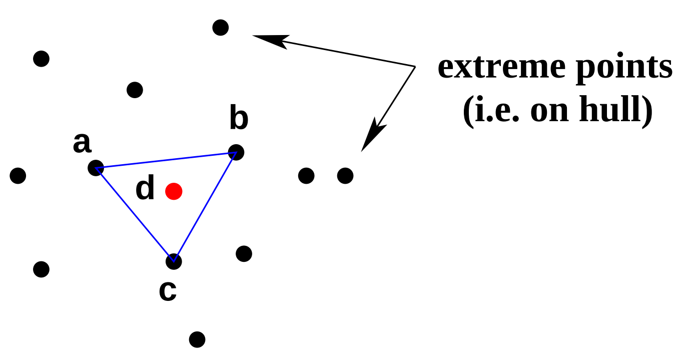
• Informally, the time complexity measures how the number of computational steps scales with the size of the problem.

• For example, an algorithm to sort n numbers in order of increasing size might have time complexity $O(n^2)$.

• Formally, this is a performance measure of the worst-case asymptotic complexity.

CH Algorithm I: Naïve extraction of extreme points 3.6

• Identify non-extreme points by finding those points inside triangles formed from the points.
• This really *is* naïve. It is included only to give an example of a very slow algorithm!



For each a of n points do (n)
  For each $b \neq a$ do $(n-1)$
    For each $c \neq b \neq a$ do $(n-2)$
      For each $d \neq c \neq b \neq a$ do $(n-3)$
        If point d is inside the triangle abc
        then d is non-extreme. (fixed time)

• Complexity is ....

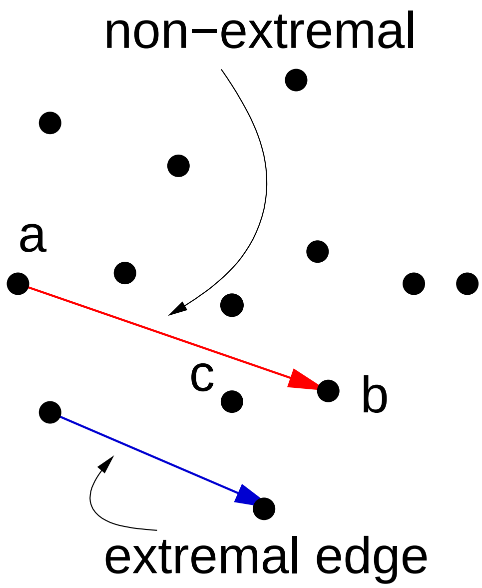
CH Algorithm II: Extreme edges

• Identify non-extreme edges by finding those that have points on both sides.

• Adopt the convention that "left" of a directed line is inside.

For each a do
    For each $b \neq a$ do
        For each $c \neq a \neq b$ do
            If point c is **not** to the left of
            line ab then line ab is non-extreme.

• Complexity is $O(n^3)$.

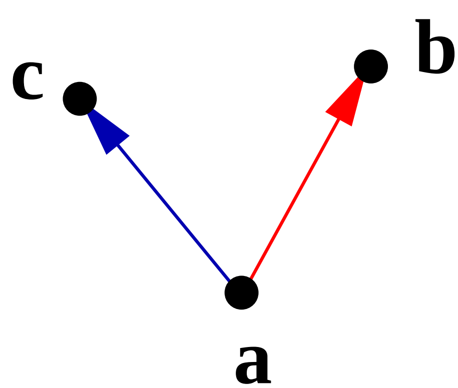


Which side of the line ab is the point c?

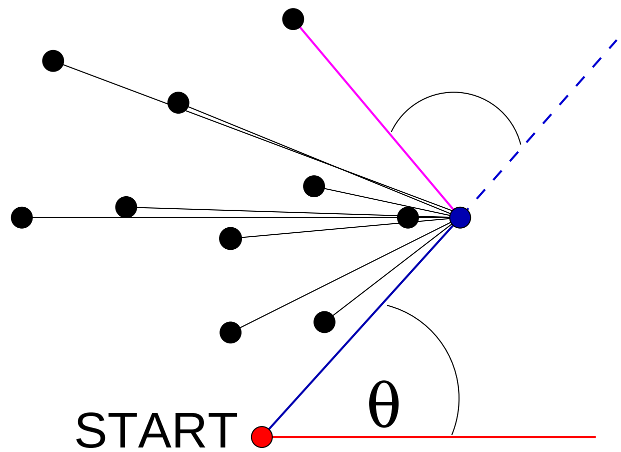
Treat **a**, **b**, **c** as three vectors $\mathbf{a} = (a_x, a_y, 0)$, etc. Use the sign of the z component of the vector product $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$.

This is

$$\begin{aligned} & \hat{\mathbf{z}} \cdot ((\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a})) \\ = & (a_x b_y - b_x a_y) + (b_x c_y - c_x b_y) + (c_x a_y - a_x c_y) \\ = & \begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ 1 & 1 & 1 \end{vmatrix} \end{aligned}$$



CH Algorithm III: "Gift wrapping"



• To start, pick the "lowest-y" point. Set a horizontal through it, then fan round, finding the point with the smallest azimuthal angle.

• Set this as the first hull edge, and use it as an anchor to find the next one.

• Note that the output is an **ordered** hull boundary.

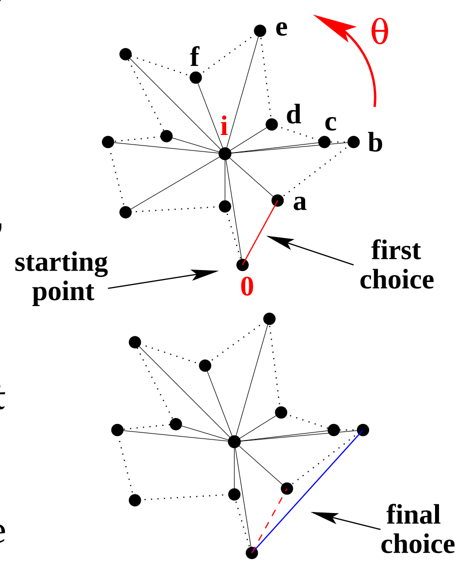
For each successive points a found on convex hull
    For each point $b \neq a$ do
        Find the point b^* with smallest angle θ from the previous edge.
        Assign b^* to the convex hull

• Complexity is ....

CH Algorithm IV: Graham's (1972)

• Sort points counter clockwise about an interior point, and grow a convex polygon.

1. Select an interior point (does not have to be!)
2. Perform an angular sort around the interior point i.
3. Perform a scan to incrementally grow the boundary, removing non-extreme points as they are detected.



• Complexity is $O(nlogn)$.
(The sort is $O(nlogn)$ and the scan complexity is at most an **addition** of $O(2n)$).

It can be shown that $nlogn$ is a lower bound on the complexity of computing a convex hull.

Representation of plane curves 3.11

A curve can be represented in **implicit** $f(x, y) = 0$ or **explicit** $y = y(x)$ or $x = x(y)$ form.

Example 1: A line in **implicit** form is

$$ax + by + c = 0.$$

Then y as an explicit function of x is

$$y = -\frac{ax}{b} - \frac{c}{b} = mx + c$$

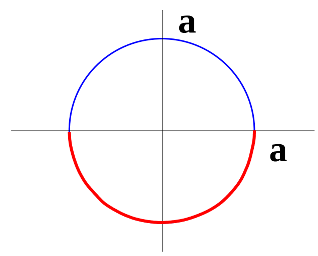
and x as an explicit function of y is

$$x = -\frac{by}{a} - \frac{c}{a}$$

Example 2: A circle is represented implicitly as

$$x^2 + y^2 = a^2$$

To obtain y **explicitly** as a function of x requires rearrangement: $y = \pm\sqrt{a^2 - x^2}$



$y = +\sqrt{a^2 - x^2}$ $y = -\sqrt{a^2 - x^2}$

Problems, problems ... 3.12

There are obvious disadvantages in using explicit forms $y = f(x)$ for curve drawing in graphics applications:

- Curves have to be drawn in several segments, e.g. $y = +\sqrt{a^2 - x^2}$ and $y = -\sqrt{a^2 - x^2}$.
- Curves cannot always be written explicitly as a function of x, e.g. a circle cannot be written as $y = f(x)$ for $x > 0$.

The implicit form does not have these problems, but

- It does not directly provide points on the curve.

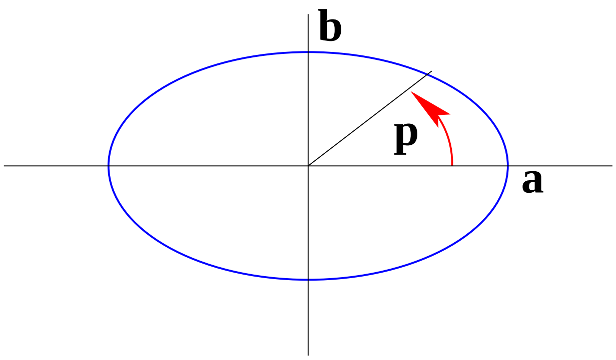
An alternative is to parameterize both x and y.

Parametric form
3.13

x and y are both functions of a single parameter p, say:

$$x = x(p) \quad y = y(p)$$

1. A line $\mathbf{x}(p) = \mathbf{a} + p\mathbf{d}$
2. An ellipse $x = a\cos p,\ y = b\sin p$.



- The parametrization is not unique, e.g. $x = a\cos p^2,\ y = b\sin p^2$ and $x = a\cos 2p,\ y = b\sin 2p$ define the same ellipse.

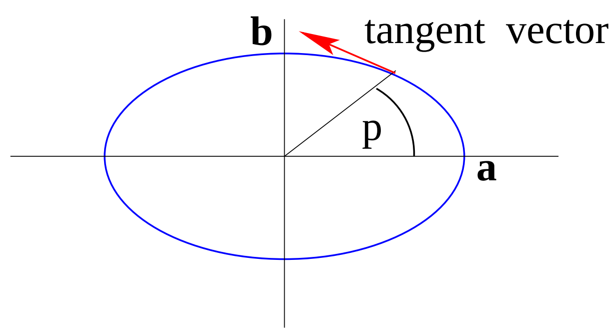
Choice of parameters
3.14

• The parameter p may be time, or by a suitable choice distance ("arc length") travelled along the curve.
• The parameter can be eliminated to obtain the implicit form: e.g. from $x = a\cos p, y = b\sin p$

$$\begin{aligned} 1 &= \cos^2 p + \sin^2 p \\ &= (x/a)^2 + (y/b)^2 \end{aligned}$$

• The derivative of the curve wrt the parameter is the tangent vector to the curve: e.g. for $x = a\cos p, y = b\sin p$

$$\begin{pmatrix} x'(p) \\ y'(p) \end{pmatrix} = \begin{pmatrix} -a\sin p \\ b\cos p \end{pmatrix}$$



Arc length s is special ... 3.15

Arc length s is special because, using Pythagoras' theorem on a short piece of planar curve, $ds = |d\mathbf{r}| = \sqrt{dx^2 + dy^2}$, *whatever the parameter p is*.

So if a curve is parameterized in terms of p as $x(p)$ and $y(p)$:

$$\frac{ds}{dp} = \sqrt{\left[\frac{dx}{dp}\right]^2 + \left[\frac{dy}{dp}\right]^2} \; .$$

So only if $\sqrt{x'(p)^2 + y'(p)^2} = 1$ is the parameter p actually the arc length.

If the parameter really is arc length, then $|d\mathbf{r}/ds| = 1$, i.e. the tangent is a unit vector.

Exercise (A) 3.16

• Describe the **3D** curve with the parametric representation

$$X = a\cos p \quad Y = a\sin p \quad Z = bp$$

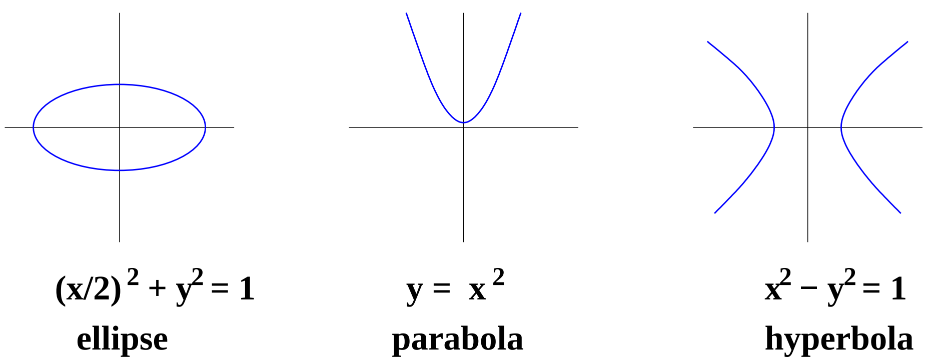
Is p arc length? How would you write p in terms of s?

Exercise (B) 3.17

• Find the implicit form of

$$x = a\frac{2s}{1+s^2} \qquad y = b\frac{1-s^2}{1+s^2} \qquad \text{with } -1 \le s \le 1$$

Conics 3.18



All conics can be represented by the implicit form:

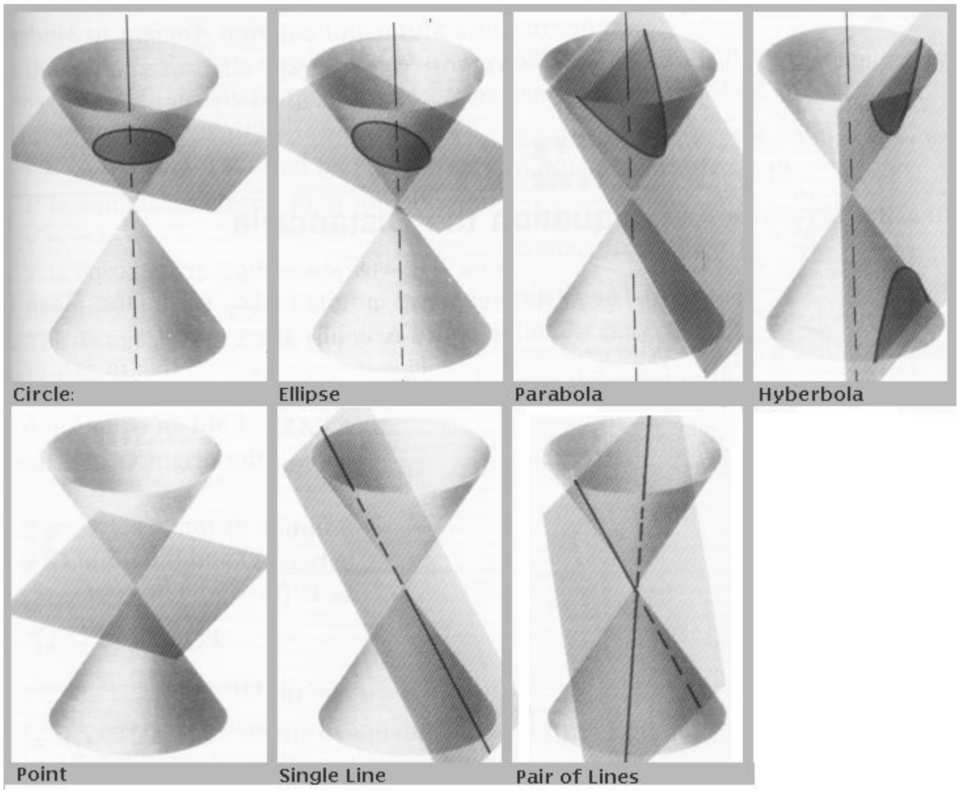
$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

i.e. a polynomial of degree two.

• A conic has five degrees of freedom in general.

Exercise What is the implicit form of a circle?

Conic sections



Conic computation

Problem: Determine the conic passing through five points.

Each point places one constraint on the conic coefficients, since if the conic passes through (x_1, y_1) then:

$$a{x_1}^2 + bx_1y_1 + c{y_1}^2 + dx_1 + ey_1 + f = 0$$

This can be written as

$$\left(\begin{array}{cccccc} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \end{array} \right) \mathbf{c} = 0$$

where $\mathbf{c}$ is the 6-vector

$$\mathbf{c} = (a, b, c, d, e, f)^\top$$

Stacking the constraints from five points

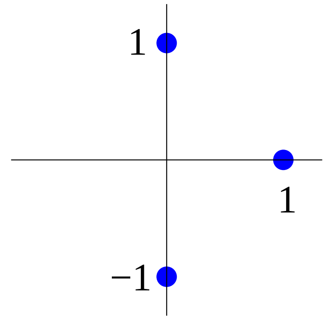
$$\left[\begin{array}{cccccc} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{array} \right] \mathbf{c} = \mathbf{0}$$

and the conic is the null-space (kernel) of this 5×6 matrix.

Example 3.21

Compute the circle through the three points $(0, 1), (1, 0), (0, -1)$.
The conic has the special form

$$a(x^2 + y^2) + dx + ey + f = 0$$



Stacking the constraints from three points

$$\begin{bmatrix} x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{bmatrix} \begin{pmatrix} a \\ d \\ e \\ f \end{pmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix} \begin{pmatrix} a \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and the circle is the null-space (kernel) of this 3×4 matrix, which is $(1, 0, 0, -1)$, i.e. $x^2 + y^2 - 1 = 0$.

Homogeneous representation 3.22

A conic $ax^2 + bxy + cy^2 + dx + ey + f = 0$ can be represented by a symmetric 3×3 matrix $\mathtt{C}$

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

or more concisely as $\mathbf{x}^\top \mathtt{C} \mathbf{x} = 0$ where $\mathbf{x} = (x, y, 1)^\top$.

Example: The ellipse $x^2/4 + y^2 = 1$ is represented by the matrix

$$\mathtt{C} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

- The conic matrix is homogeneous (as is any implicit form), since multiplying by a constant does not affect the curve.
- The homogeneous representation is particularly useful for transforming conics.
- If the conic matrix has rank less than three then the conic is degenerate.

Conic Shadows

Ex1: Find the shape of the illuminated region cast by a (circular) torch on a plane

Ex2: Find the shape of the illuminated region cast by the table lamp on the wall.

Compute the curve on the plane either by

- Intersecting the light cone with a plane, or by
- Projectively transforming a circle from one plane to another.

Transforming conics

Under a plane to plane transformation $\mathbf{x}' = \mathtt{H}\mathbf{x}$, a conic transforms as

$$\mathtt{C}' = \mathtt{H}^{-\top}\mathtt{C}\mathtt{H}^{-1}$$

where $\mathbf{x} = (x, y, 1)^\top$, $\mathbf{x}' = (x', y', 1)^\top$, $\mathtt{H}$ is a homogeneous 3×3 matrix, and $\mathtt{H}^{-\top} = (\mathtt{H}^{-1})^\top$

Proof

Start from

$$\mathbf{x}^\top \mathtt{C}\mathbf{x} = 0$$

If $\mathbf{x}' = \mathtt{H}\mathbf{x}$, then $\mathbf{x} = \mathtt{H}^{-1}\mathbf{x}'$, and substituting for $\mathbf{x}$

$$\begin{aligned} \mathbf{x}^\top \mathtt{C}\mathbf{x} &= \mathbf{x}'^\top [\mathtt{H}^{-1}]^\top \mathtt{C}\mathtt{H}^{-1}\mathbf{x}' \\ &= \mathbf{x}'^\top \mathtt{H}^{-\top}\mathtt{C}\mathtt{H}^{-1}\mathbf{x}' = 0 \end{aligned}$$

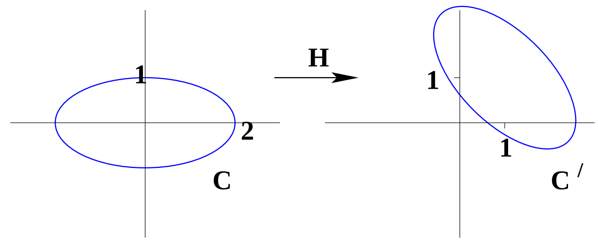
which is a quadratic form $\mathbf{x}'^\top \mathtt{C}'\mathbf{x}' = 0$ with

$$\mathtt{C}' = \mathtt{H}^{-\top}\mathtt{C}\mathtt{H}^{-1}$$

and, again, is a symmetric matrix which represents a conic.

Example 3.25

Determine the conic representing the ellipse $x^2 + 4y^2 = 4$ after a clockwise rotation of 45^o and a translation of $(1, 1)$.



$$\mathtt{R}(-45^\circ) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\mathtt{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix} \qquad \mathtt{H} = \left[\begin{array}{c|c} \mathtt{R}(-45) & \begin{matrix} 1 \\ 1 \end{matrix} \\ \hline 0 \quad 0 & 1 \end{array}\right] \qquad \mathtt{H}^{-1} = \left[\begin{array}{c|c} \mathtt{R}(45) & \begin{matrix} 0 \\ -\sqrt{2} \end{matrix} \\ \hline 0 \quad 0 & 1 \end{array}\right]$$

$$\mathtt{C}' = \mathtt{H}^{-\top}\mathtt{C}\mathtt{H}^{-1} = \left[\begin{array}{c|c} \mathtt{R}(-45) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 \quad -\sqrt{2} & 1 \end{array}\right] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix} \left[\begin{array}{c|c} \mathtt{R}(45) & \begin{matrix} 0 \\ -\sqrt{2} \end{matrix} \\ \hline 0 \quad 0 & 1 \end{array}\right] = \begin{bmatrix} 5/2 & 3/2 & -4 \\ 3/2 & 5/2 & -4 \\ -4 & -4 & 4 \end{bmatrix}$$

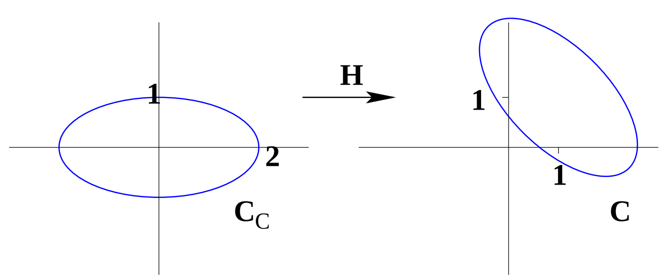
This is the conic $\quad 2.5x^2 + 3xy + 2.5y^2 - 8x - 8y + 4 = 0.$

Conic drawing problem 3.26

How can we draw the conic

$$2.5x^2 + 3xy + 2.5y^2 - 8x - 8y + 4 = 0$$

1. Determine the canonical form and the Euclidean transformation to the canonical frame
2. Parametrize the canonical form. This determines points on the conic.
3. Map points back to original frame.



Here $\mathtt{C}$ is known, but the canonical form $\mathtt{C}_\mathrm{C}$ and Euclidean transformation $\mathtt{H}$ are unknown.

But, if $\mathtt{C} = \mathtt{H}^{-\top}\mathtt{C}_\mathrm{C}\mathtt{H}^{-1}$, then $\mathtt{C}_\mathrm{C} = \mathtt{H}^\top\mathtt{C}\mathtt{H}$.

In fact, we are going to take it in two steps.

1a) The rotation is an eigen problem 3.27

- The matrix of the conic we desire to draw is

$$\mathtt{C} = \begin{bmatrix} 5/2 & 3/2 & -4 \\ 3/2 & 5/2 & -4 \\ -4 & -4 & 4 \end{bmatrix} = \begin{bmatrix} \mathtt{C}_{\text{topleft } 2\times 2} & \mathbf{d} \\ \mathbf{d}^\top & \delta \end{bmatrix}$$

- Diagonalize top-left 2×2. From this $\mathtt{C}_{\text{topleft } 2\times 2} = \mathtt{R}\Lambda\mathtt{R}^\top$

$$\begin{bmatrix} 5/2 & 3/2 \\ 3/2 & 5/2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

where $\mathtt{R}$ transforms **points** on an ellipse A aligned with the canonical one to the desired one. That is $\mathbf{x} = \mathtt{H}_{\text{rot}}\mathbf{x}_A = \begin{bmatrix} \mathtt{R} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}\mathbf{x}_A$.

So that means $\mathbf{x}^\top\mathtt{C}\mathbf{x} = \mathbf{x}_A^\top\mathtt{H}_{\text{rot}}^\top\mathtt{C}\mathtt{H}_{\text{rot}}\mathbf{x}_A = \mathbf{x}_A^\top\mathtt{C}_A\mathbf{x}_A = 0$

Also note that $\mathtt{C}_A$ can be written as $\mathtt{C}_A = \begin{bmatrix} \Lambda & \mathbf{a} \\ \mathbf{a}^\top & \alpha \end{bmatrix} = \begin{bmatrix} \Lambda & \mathtt{R}^\top\mathbf{d} \\ \mathbf{d}^\top\mathtt{R} & \delta \end{bmatrix}$

1b Translate to complete the diagonalization. 3.28

Now we want to find the translation $\mathbf{t}$ such that $\mathbf{x}_A = \mathtt{H}_{\text{trans}}\mathbf{x}_C = \begin{bmatrix} \mathtt{I} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}\mathbf{x}_C$

But $\quad \mathbf{x}_A^\top\mathtt{C}_A\mathbf{x}_A = \mathbf{x}_C^\top\mathtt{H}_{\text{trans}}^\top\mathtt{C}_A\mathtt{H}_{\text{trans}}\mathbf{x}_C = \mathbf{x}_C^\top\mathtt{C}_C\mathbf{x}_C \quad$ So

$$\begin{aligned} \mathtt{C}_C &= \mathtt{H}_{\text{trans}}^\top\mathtt{C}_A\mathtt{H}_{\text{trans}} \\ &= \begin{bmatrix} \mathtt{I} & \mathbf{0} \\ \mathbf{t}^\top & 1 \end{bmatrix}\begin{bmatrix} \Lambda & \mathbf{a} \\ \mathbf{a}^\top & \alpha \end{bmatrix}\begin{bmatrix} \mathtt{I} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} = \begin{bmatrix} \Lambda & \Lambda\mathbf{t} + \mathbf{a} \\ \mathbf{t}^\top\Lambda + \mathbf{a}^\top & \mathbf{t}^\top\Lambda\mathbf{t} + 2\mathbf{a}.\mathbf{t} + d \end{bmatrix} = \begin{bmatrix} \Lambda & \mathbf{0} \\ \mathbf{0}^\top & c \end{bmatrix} \end{aligned}$$

The key thing to emerge is that

$$\mathbf{t} = -\Lambda^{-1}\mathbf{a} = -\Lambda^{-1}\mathtt{R}^\top\mathbf{d}$$

Hence in our example

$$\mathbf{t} = -\begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} -4 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}$$

Now work out the canonical ellipse matrix ... 3.29

$$\begin{aligned}
\mathtt{C}_{\mathrm{C}} &= \mathtt{H}_{\mathrm{trans}}^{\top}\mathtt{C}_{\mathrm{A}}\mathtt{H}_{\mathrm{trans}} \\
&= \mathtt{H}_{\mathrm{trans}}^{\top}\mathtt{H}_{\mathrm{rot}}^{\top}\mathtt{C}\mathtt{H}_{\mathrm{rot}}\mathtt{H}_{\mathrm{trans}} \\
&= \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & \sqrt{2} & 1 \end{array}\right]\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 4 & -4\sqrt{2} \\ 0 & -4\sqrt{2} & 4 \end{array}\right]\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & \sqrt{2} \\ \hline 0 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{array}\right]
\end{aligned}$$

So now "draw" the canonical conic $x_{\mathrm{C}}^2 + 4y_{\mathrm{C}}^2 = 4$ using the parametrization

$$x_{\mathrm{C}}(t) = 2\cos t \qquad y_{\mathrm{C}}(t) = \sin t \quad 0 \le t < 2\pi$$

... and transform each point into the desired frame 3.30

Then the composite transformation is by a homogeneous matrix

$$\begin{aligned}
\mathbf{x}(t) &= \mathtt{H}_{\mathrm{rot}}\mathtt{H}_{\mathrm{trans}}\begin{pmatrix} x_C \\ y_C \\ 1 \end{pmatrix} \\
&= \mathtt{H}_{\mathrm{rot}}\mathtt{H}_{\mathrm{trans}}\begin{pmatrix} 2\cos t \\ \sin t \\ 1 \end{pmatrix} \\
&= \left[\begin{array}{cc|c} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right]\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & \sqrt{2} \\ \hline 0 & 0 & 1 \end{array}\right]\begin{pmatrix} 2\cos t \\ \sin t \\ 1 \end{pmatrix}
\end{aligned}$$

Plane curve intersections 3.31

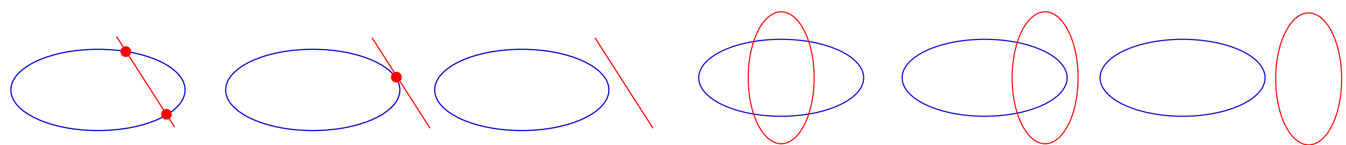
A common problem in graphics is that of finding the intersection(s) of two plane curves.

Bézout's Theorem says that *There are at most mn intersections between two plane curves of algebraic degree m and n.*

Examples

1. Line $ax + by + c$ (degree 1)
ellipse $x^2/4 + y^2 = 1$ (degree 2).
There are at most $1 \times 2 = 2$ intersections.

Between two ellipses there are $2 \times 2 = 4$ intersections.



Computing intersections 3.32

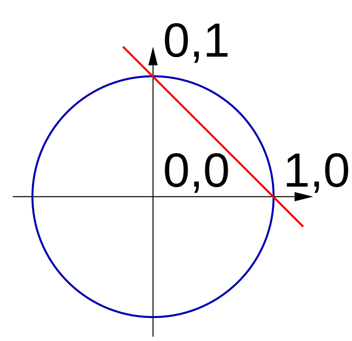
• Usually it is best to use one implicit form and one parametrized.

Example:

Compute the intersections between the circle $x^2 + y^2 - 1 = 0$ and the line $x + y - 1 = 0$.
1. Parametrize the line: $x = t, y = 1 - t$.
2. Substitute into the circle implicit equation:

$$\begin{aligned} x^2 + y^2 - 1 &= t^2 + (1-t)^2 - 1 \\ &= t^2 + 1 - 2t + t^2 - 1 \\ &= 2t^2 - 2t \\ &= 2t(t-1) = 0 \end{aligned}$$

3.The solutions are $t = 0$ or $t = 1$.
4. With $t = 0$, the point is $(0, 1)$; with $t = 1$, the point is $(1, 0)$.



Quadrics 3.33

• A quadric is a quadratic surface in 3D.

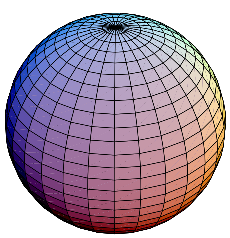
• It is represented by a symmetric 4×4 matrix $\mathtt{Q}$ as

$$\mathbf{X}^\top \mathtt{Q} \mathbf{X} = 0$$

with $\mathbf{X} = (X, Y, Z, 1)^\top$.

Examples of Quadrics 3.34

Sphere centred at origin
$X^2 + Y^2 + Z^2 = 1$

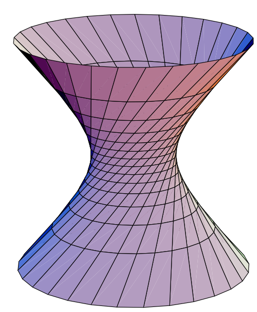


$$\mathtt{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Cylinder along Z axis
$X^2 + Y^2 = 1$

$$\mathtt{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Hyperboloid of Revolution
$X^2 + Y^2 = Z^2 + 1$



$$\mathtt{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Quadrics, more 3.35

• Similar ideas (parametrizations, homogeneous transformations, classifications etc) apply to quadrics

• Just with one dimension more than conics.

• There are more canonical cases due to the extra dimension.