ON SOME FUNCTIONAL EQUATION GENERALIZING
CAUCHY'S AND D'ALEMBERT'S FUNCTIONAL EQUATIONS

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0. Introduction. Let $G$ be a compact group and $\Gamma$ a locally compact Abelian group. Suppose $\Gamma$ is a $G$-space in the sense that there is a map

$$G \times \Gamma \ni (g, x) \mapsto gx \in \Gamma$$

such that $g(hx) = (gh)x$, $ex = x$, and $g(x+y) = gx + gy$ for all $g, h \in G$ and all $x, y \in \Gamma$, $e$ being the neutral element of $G$. Suppose, moreover, that the action of $G$ is measurable in the sense that the function $G \times \Gamma \ni (g, x) \mapsto gx \in \Gamma$ and all the functions $G \times \Gamma \ni (g, y) \mapsto gx + y \in \Gamma$ ($x \in \Gamma$) are $(m \times \mu, \mu)$-measurable, where $m$ and $\mu$ are Haar measures on $G$ and $\Gamma$, respectively.

The aim of this paper is to study $\mu$-measurable, in various senses essentially bounded solutions to the functional equation

$$\int G f(x+gy) \, dm(g) = f(x)f(y) \quad (x, y \in \Gamma). \quad (0.1)$$

Two particular cases of this equation have already been the object of investigations:

1° Cauchy’s functional equation

$$f(x+y) = f(x)f(y) \quad (x, y \in \Gamma)$$

resulting from (0.1) upon taking $G = \mathbb{Z}_1$, with normalized Haar measure, to act trivially on $\Gamma$;

2° d’Alembert’s functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y) \quad (x, y \in \Gamma)$$

resulting from (0.1) if $G = \mathbb{Z}_2$, with normalized Haar measure, is taken to act on $\Gamma$ by the rule $0 \cdot x = x$, $1 \cdot x = -x$ ($x \in \Gamma$).
In either of these cases an arbitrary group $\Gamma$ is a $G$-space for a group $G$ of a particularly simple form. Specific groups $\Gamma$ can apparently be realized as $G$-spaces for more complicated groups $G$. A simple example is furnished by the additive group of complex numbers $C$ which is a $Z_k$-space under the action

\[(g, x) \rightarrow \left[ \exp \left( \frac{2\pi i}{k} g \right) \right] x \quad (k \in N),\]

and is also a $T$-space if the circle group $T$ is taken to act on $C$ by multiplication.

Our subsequent considerations will proceed in the following order. First, we find all $\mu$-measurable $\mu$-essentially bounded and $\mu$-essentially non-zero solutions of the equation in question. Next, we discuss generalized complex-valued solutions in the case where $G$ is finite and $\Gamma$ is discrete; the identity (0.1) in this case is only assumed to hold almost everywhere with respect to certain translation-invariant ideals of subsets of $\Gamma$. Finally, we study some generalized solutions whose values lie in commutative semi-simple Banach algebras. As a result in this last case we obtain a generalization of a theorem of de Bruijn [2] on almost additive functions.

It should be pointed out that none of our results will essentially depend on a particular choice of the norming constant $m(G)$. This is due to the fact that a function $f$ satisfies (0.1) if and only if the function $cf (c > 0)$ satisfies the same identity with $m$ replaced by $cm$.

1. Solutions in $L^\infty (\Gamma)$. Let $L^\infty (\Gamma)$ be the space of all complex $\mu$-measurable $\mu$-essentially bounded functions on $\Gamma$, $\hat{\Gamma}$ the dual group of $\Gamma$.

Given $\chi \in \hat{\Gamma}$, put

\[(1.1) \quad f(x) = \int_G (gx, \chi) dm(g) \quad (x \in \Gamma).\]

Clearly, $f$ is in $L^\infty (\Gamma)$. We claim that $f$ satisfies (0.1). In fact, if $x, y \in \Gamma$, then

\[
\int_{\Gamma} f(x + gy) dm(g) = \int_G \left[ \int_{\Gamma} (hx + hgy, \chi) dm(h) \right] dm(g)
= \int_G (hx, \chi) \left[ \int_{\Gamma} (hgy, \chi) dm(g) \right] dm(h)
= \int_G (hx, \chi) \left[ \int_{\Gamma} (gy, \chi) dm(g) \right] dm(h)
= f(x)f(y).
\]

Conversely, we have the following

**Theorem 1.1.** Any $\mu$-essentially non-zero function in $L^\infty (\Gamma)$ satisfying (0.1) can be represented in the form (1.1) for some $\chi \in \hat{\Gamma}$.
This theorem is a particular case of the following more general result:

**Theorem 1.2.** Let \( w \) be a \( \mu \)-essentially non-zero function in \( L^\infty(\Gamma) \), \( \Omega \) a subset of \( \Gamma \), and \( f \) a complex function on \( \Omega \). Suppose that for each \( y \in \Omega \) the identity

\[
\int_G w(x + gy) \, dm(g) = f(y) w(x)
\]

holds for \( \mu \)-almost all \( x \) in \( \Gamma \). Then there exists \( \chi \in \hat{\Gamma} \) such that for every \( y \in \Omega \)

\[
f(y) = \int_G (gy, \chi) \, dm(g).
\]

**Proof.** Let \( A(\hat{\Gamma}) \) be the space of Fourier transforms of functions in \( L^1(\Gamma) \) with the norm \( \| \varphi \|_{A(\hat{\Gamma})} = \| u \|_1 \), where \( \varphi = \hat{u} \); we adopt the following convention as regards the Fourier transform:

\[
\hat{u}(\chi) = \int_G \overline{u(x)(x, \chi)} \, d\mu(x) \quad (\chi \in \hat{\Gamma}).
\]

Given \( y \in \Omega \), define the function \( \varepsilon_y \) on \( \hat{\Gamma} \) to be

\[
\varepsilon_y(\chi) = \int_G (gy, \chi) \, dm(g) \quad (\chi \in \hat{\Gamma}).
\]

Note that if \( \varphi \in A(\hat{\Gamma}) \), then \( \varepsilon_y \varphi \in A(\hat{\Gamma}) \). In fact, if \( \varphi = \hat{u} \), then \( \varepsilon_y \varphi \) is the Fourier transform of the function

\[
x \to \int_G u(x + gy) \, dm(g) \quad (x \in \Gamma),
\]

an element of \( L^1(\Gamma) \).

Let \( \hat{w} \) be the Fourier transform of \( w \) regarded as a pseudomeasure on \( \hat{\Gamma} \) (cf. [1]), i.e., \( \hat{w} \) is the linear continuous functional on \( A(\hat{\Gamma}) \) defined by

\[
\langle \hat{w}, \varphi \rangle = \int_G w(-x) u(x) \, d\mu(x) \quad (\varphi \in A(\hat{\Gamma}), \varphi = \hat{u}).
\]

We claim that for each \( y \in \Omega \) and each \( \varphi \in A(\hat{\Gamma}) \)

\[
[(\varepsilon_y - f(y)) \varphi] \hat{w} = 0.
\]

(1.2)

In fact, for each \( \psi \in A(\hat{\Gamma}) \) we have

\[
\langle [\varepsilon_y - f(y)] \varphi, \psi \rangle = \langle \hat{w}, \varepsilon_y \varphi \psi \rangle - f(y) \langle \hat{w}, \varphi \psi \rangle = \int_G w(-x) \left[ \int_G (u \ast v)(x + gy) \, dm(g) \right] \, d\mu(x)
\]

\[
- f(y) \int_G w(-x) (u \ast v)(x) \, d\mu(x)
\]

\[
= \int_G (u \ast v)(-x) \left[ \int_G w(x + gy) \, dm(g) \right] \, d\mu(x)
\]

\[
- f(y) \int_G w(x)(u \ast v)(-x) \, d\mu(x) = 0,
\]

where \( \varphi = \hat{u}, \psi = \hat{v} \), and \( \ast \) stands for convolution.
Since \( w \) is \( \mu \)-essentially non-zero, the support of \( \hat{w} \) is non-void. Let \( \chi \) be a point in the support of \( \hat{w} \). From (1.2) we infer that for every \( y \in \Omega \) and every \( \varphi \in A(\hat{F}) \) the function \( (e_y - f(y)) \varphi \) vanishes at \( \chi \) (cf. [1], Theorem 1.3.1). Consequently, for all \( y \in \Omega \) we have \( e_y(\chi) = f(y) \), which is the desired representation.

2. Generalized complex-valued solutions. We begin by recalling certain concepts and introducing some notation.

An ideal of subsets of a set \( X \) is a family \( \mathfrak{I} \) of subsets of \( X \) such that
(i) \( \emptyset \in \mathfrak{I} \) and \( X \notin \mathfrak{I} \);
(ii) if \( A \in \mathfrak{I} \) and \( B \subset A \), then \( B \in \mathfrak{I} \);
(iii) if \( A, B \in \mathfrak{I} \), then \( A \cup B \in \mathfrak{I} \).
Here the condition that \( X \notin \mathfrak{I} \) is not standard.

Let \( X \) be a set and \( \mathfrak{I} \) an ideal of subsets of \( X \). If \( f \) is a real function on \( X \), the \( \mathfrak{I} \)-essential supremum of \( f \) is defined as

\[
\sup_\mathfrak{I} f = \inf \{ c \in \mathbb{R} : f^{-1}(\{c, +\infty\}) \in \mathfrak{I} \}.
\]

Two complex functions \( f \) and \( g \) on \( X \) are equal \( \mathfrak{I} \)-almost everywhere (in symbol, \( f =_\mathfrak{I} g \)) if and only if \( \sup_\mathfrak{I} |f-g| = 0 \).

Let \( l_3^\infty(X) \) be the algebra of all complex functions \( f \) on \( X \) such that

\[
\sup_\mathfrak{I} |f| < +\infty.
\]

\( \|f\|_3 = \sup_\mathfrak{I} |f| \) is a pseudonorm on \( l_3^\infty(X) \), \( N_3(X) = \{ f \in l_3^\infty(X) : \|f\|_3 = 0 \} \) is an ideal of \( l_3^\infty(X) \), and the quotient algebra \( l_3^\infty(X) = l_3^\infty(X)/N_3(X) \) with the induced norm is, as one easily verifies, a Banach algebra with unit. The canonical image in \( l_3^\infty(X) \) of a function \( f \in l_3^\infty(X) \) will be denoted by \([f]_3\).

If \( A \) is a subset of \( X \) and \( f \) a function on \( X \), then \( f|_A \) stands for the restriction of \( f \) to \( A \).

Let \( H \) be a discrete Abelian group. Given a function \( f \) on \( H \) and \( x \in H \), \( T_x f \) denotes the translate of \( f \) by \( x \).

An ideal \( \mathfrak{I} \) of subsets of \( H \) is translation-invariant if \( A \in \mathfrak{I} \) implies \( A+x \in \mathfrak{I} \) for all \( x \in H \), where

\[
A+x = \{ y \in H : y = a+x, \ a \in A \}.
\]

If \( \mathfrak{I} \) is a translation-invariant ideal of subsets of \( H \), \( f \in l_3^\infty(H) \), and \( x \in H \), then the element \( T_x([f])_3 \) in \( l_3^\infty(H) \) is well-defined as the class \([T_x f]_3\), and we have

\[
\|T_x[f]_3\|_3 = \|T_x([f])_3\|_3.
\]

Given a linear continuous functional \( \xi \) on \( l_3^\infty(H) \) and \( x \in H \), \( T_x \xi \) stands for the linear continuous functional on \( l_3^\infty(H) \) defined by

\[
T_x \xi([f])_3 = \xi(T_x([f])_3) \quad (f \in l_3^\infty(H));
\]

we have, of course, \( \|T_x \xi\| = \|\xi\| \).
For the remainder of the paper, we will assume that the group $G$ is finite and the group $\Gamma$ is discrete. The conditions for the action of $G$ to be measurable will be now vacuous.

**Theorem 2.1.** Let $\mathcal{I}$ be a translation-invariant ideal of subsets of $\Gamma$, $\Omega$ a subset of $\Gamma$, $f$ a function in $L^\infty_\mathcal{I}(\Gamma)$ such that $\|f\|_\mathcal{I} \neq 0$ and for all $y \in \Omega$

$$
\sum_{g \in G} f(\cdot + gy) = \mathcal{I} f(\cdot) f(y).
$$

Then there exists a complex bounded function $F$ on $\Gamma$ such that

$$
\sum_{g \in G} F(x + gy) = F(x) F(y)
$$

for all $x, y \in \Gamma$ and $F|_{\Omega} = f|_{\Omega}$.

**Proof.** Let $\xi$ be a linear continuous functional on $L^\infty_\mathcal{I}(\Gamma)$ such that $\xi([f]_\mathcal{I}) \neq 0$. In view of (2.1), we have for all $y \in \Omega$ and all $z \in \Gamma$

$$
\sum_{g \in G} T_z \xi([T_{gy}f]_\mathcal{I}) = f(y) T_z \xi([f]_\mathcal{I}),
$$

whence

$$
\sum_{g \in G} T_{z + gy} \xi([f]_\mathcal{I}) = f(y) T_z \xi([f]_\mathcal{I}).
$$

Letting $w$ denote the function $x \rightarrow T_x \xi([f]_\mathcal{I})$, we have for all $y \in \Omega$

$$
\sum_{g \in G} T_{gy} w = f(y) w.
$$

By Theorem 1.2, there exists $\chi$ in $\hat{\Gamma}$ such that

$$
f(y) = \sum_{g \in G} (gy, \chi)
$$

for all $y \in \Omega$. To complete the proof, it suffices to take $F$ to be the function

$$
x \rightarrow \sum_{g \in G} (gx, \chi).
$$

The function $F$ in the statement of Theorem 2.1 is in general not unique. Below we present a theorem assuring the uniqueness of $F$ in the case where $\Gamma \backslash \Omega$ is an element of a translation-invariant ideal of subsets of $\Gamma$.

**Theorem 2.2.** Let $\mathcal{I}$ be a translation-invariant ideal of subsets of $\Gamma$, and $F$ a complex bounded function on $\Gamma$ satisfying (2.2) for all $x, y \in \Gamma$. Then $F$ is the only complex bounded function in $[F]_\mathcal{I}$ satisfying (2.2) for all $x, y \in \Gamma$. Moreover, $\|F\|_\infty = \|F\|_\mathcal{I}$.

**Proof.** Let $b\Gamma$ be the Bohr compactification of $\Gamma$, $\alpha$ the canonical homomorphism from $\Gamma$ into $b\Gamma$, $AP(\Gamma)$ the algebra of all complex almost
periodic functions on \( \Gamma \), and \( C(b\Gamma) \) the algebra of all complex continuous functions on \( b\Gamma \). As is known, the mapping

\[
C(b\Gamma) \ni h \mapsto h \circ \alpha \in AP(\Gamma)
\]

is a Banach algebra isomorphism.

Let \( \xi \) be a linear multiplicative functional on \( l^{\infty}_\mathcal{I}(\Gamma) \). The mapping \( h \mapsto \xi([h \circ \alpha]_\mathcal{I}) \) is a linear multiplicative functional on \( C(b\Gamma) \) and as such it can be represented in the form \( h \mapsto h(\omega) \) for some \( \omega \in b\Gamma \). If \( \omega \in b\Gamma \) and \( h \in C(b\Gamma) \), then

\[
h(\omega) = \lim_{\alpha(x) \to \omega - \omega_y} T_x \xi([h \circ \alpha]_\mathcal{I}),
\]

passage to the limit being possible due to the fact that \( \alpha(\Gamma) \) is dense in \( b\Gamma \). It follows from the latter formula that every almost periodic function \( f \) on \( \Gamma \) is uniquely determined by its restriction to a subset of \( \Gamma \) whose complement belongs to \( \mathcal{I} \) and, moreover, \( \|f\|_\infty = \|f\|_\mathcal{I} \). The uniqueness of \( F \) and the identity \( \|F\|_\infty = \|F\|_\mathcal{I} \) result now from the fact that, by virtue of Theorem 1.1, any complex bounded function satisfying (2.2) for all \( x, y \in \Gamma \) is a trigonometric polynomial on \( \Gamma \).

3. Generalized Banach algebra valued solutions. Let \( A \) be a commutative semi-simple Banach algebra. Suppose \( \Phi \) is a subset of the Gelfand space of \( A \) such that the only element of \( A \) whose Gelfand transform vanishes at all points of \( \Phi \) is the zero element. Let \( s(\Phi) \) be the algebra of all complex functions on \( \Phi \), equipped with the topology of pointwise convergence. When \( \phi \in \Phi \) and \( f \in s(\Phi) \), we write \( \langle \phi, f \rangle \) for the value of \( f \) at \( \phi \). Let \( \sigma_\phi \) be the weak topology on \( A \) induced by \( \Phi \): \( \sigma_\phi \) has for a basis of neighbourhoods of the origin the sets

\[
\{x \in A: \max_{1 \leq i \leq n} |\langle \phi_i, x \rangle| < \varepsilon\}
\]

with \( \{\phi_1, \ldots, \phi_n\} \) running over all finite subsets of \( \Phi \) and \( \varepsilon \) running over all positive numbers. We identify \( A \) under \( \sigma_\phi \) with the topological subalgebra of \( s(\Phi) \) consisting of all restrictions to \( \Phi \) of the Gelfand transforms of elements of \( A \).

An immediate consequence of Theorems 2.1 and 2.2 is the following

**Theorem 3.1.** For each \( \phi \in \Phi \), let \( \mathcal{I}_\phi \) and \( \mathcal{I}_\phi \) be translation-invariant ideals of subsets of \( \Gamma \), and \( \Omega_\phi \) a subset of \( \Gamma \) such that \( \Gamma \setminus \Omega_\phi \in \mathcal{I}_\phi \). Let \( f \) be a function from \( \Gamma \) into \( A \) such that, given \( \phi \in \Phi \),

(i) the function \( x \mapsto \langle \phi, f(x) \rangle \) is in \( l^{\infty}_\mathcal{I}_\phi(\Gamma) \);

(ii) \( \|\langle \phi, f(\cdot) \rangle\|_{\mathcal{I}_\phi} = 0 \) implies \( \|\langle \phi, f(\cdot) \rangle\|_{\mathcal{I}_\phi} = 0 \);

(iii) for every \( y \in \Omega_\phi \)

\[
\sum_{g \in G} \langle \phi, f(g + gy) \rangle =_{\mathcal{I}_\phi} \langle \phi, f(\cdot) \rangle \langle \phi, f(y) \rangle.
\]

(3.1)
Then there exists a unique bounded function $F$ from $\Gamma$ into $s(\Phi)$ satisfying (2.2) for all $x, y \in \Gamma$, such that, for each $\phi \in \Phi$,

$$\langle \phi, F(\cdot) \rangle = \lambda_\phi \langle \phi, f(\cdot) \rangle.$$  

Moreover, for every $\phi \in \Phi$,

$$\|\langle \phi, F(\cdot) \rangle\|_\infty = \|\langle \phi, f(\cdot) \rangle\|_{\lambda_\phi}$$

and if $\|\langle \phi, f(\cdot) \rangle\|_{\lambda_\phi} \neq 0$, then

$$\langle \phi, F(x) \rangle = \langle \phi, f(x) \rangle \quad \text{for all } x \in \Omega_{\phi}.$$  

Specializing the hypotheses of this theorem, we shall now derive a few results relating to the situation in which exact solutions of the equation under study take values in the same Banach algebra as initial generalized solutions.

**Theorem 3.2.** Suppose the unit ball of $A$ is $\sigma_\phi$-compact. Let $\mathcal{I}$ be a translation-invariant ideal of subsets of $\Gamma$. For each $\phi \in \Phi$, let $\Omega_{\phi}$ be a subset of $\Gamma$ with $\Gamma \setminus \Omega_{\phi} \in \mathcal{I}$ and $\mathcal{I}_{\phi}$ a translation-invariant ideal of subsets of $\Gamma$. Let $f$ be a norm-bounded function from $\Gamma$ into $A$ such that, given $\phi \in \Phi$,

(i) $\|\langle \phi, f(\cdot) \rangle\|_{\lambda_\phi} = 0$ implies $\|\langle \phi, f(\cdot) \rangle\|_{\lambda} = 0$;

(ii) (3.1) holds for every $y \in \Omega_{\phi}$.

Then there exists a unique bounded function $F$ from $\Gamma$ into $(A, \sigma_\phi)$ satisfying (2.2) for all $x, y \in \Gamma$, such that, for each $\phi \in \Phi$,

$$\langle \phi, F(\cdot) \rangle = \lambda_\phi \langle \phi, f(\cdot) \rangle.$$  

$F$ is bounded in norm and, moreover, for every $\phi \in \Phi$,

$$\|\langle \phi, F(\cdot) \rangle\|_\infty = \|\langle \phi, f(\cdot) \rangle\|_{\lambda}$$

and if $\|\langle \phi, f(\cdot) \rangle\|_{\lambda_\phi} \neq 0$, then

$$\langle \phi, F(x) \rangle = \langle \phi, f(x) \rangle \quad \text{for all } x \in \Omega_{\phi}.$$  

**Proof.** In view of the preceding theorem, we need only to prove the existence part.

Let $\xi$ be a linear multiplicative functional on $l^\infty(\Gamma)$ and $\omega_\xi$ the corresponding point in $b\Gamma$ such that (2.3) holds. Since, clearly, for each $\phi \in \Phi$ the function $x \mapsto \langle \phi, f(x) \rangle$ belongs to $l^\infty(\Gamma)$ and is a trigonometric polynomial modulo $N_\mathcal{I}(\Gamma)$, the right-hand side of the formula

$$(3.2) \quad \langle \phi, F(x) \rangle = \lim_{a(y) - \omega_\xi \rightarrow 0} T_x \xi([\langle \phi, f(\cdot) \rangle]_{\mathcal{I}}) \quad (x \in \Gamma)$$

makes sense. Reading the proofs to Theorems 2.1 and 2.2 we see that the assertion will follow once we prove that the function $F$ defined by (3.2) is $A$-valued and is bounded in norm. By the compactness hypothesis, all reduces
to proving that, given \( x \in \Gamma \), \( \phi \rightarrow \langle \phi, F(x) \rangle \), an element of \( s(\Phi) \), lies in the closure of \( f(\Gamma) \).

Suppose \( a_1, \ldots, a_n \in C \) and \( \phi_1, \ldots, \phi_n \in \Phi \) satisfy

\[
|\sum_{i=1}^{n} a_i \langle \phi_i, f(y) \rangle| \leq 1
\]

for all \( y \in \Gamma \). Then, in view of (3.2),

\[
|\sum_{i=1}^{n} a_i \langle \phi_i, F(x) \rangle| \leq 1
\]

whatever \( x \in \Gamma \). Every linear continuous functional on \( s(\Phi) \) being of the form

\[
h \rightarrow \sum_{i=1}^{n} a_i \langle \phi_i, h \rangle
\]

for some \( a_1, \ldots, a_n \in C \) and some \( \phi_1, \ldots, \phi_n \in \Phi \), we reach the conclusion by utilizing the bipolar theorem.

In the sequel, if \( g \in G \) and \( \Sigma \) is a subset of \( \Gamma \), we write \( \Sigma_g \) for the set \( \{x \in \Gamma: x - gx \in \Sigma\} \); for \( x \in \Gamma \), we let

\[
x - \Sigma = \{y \in \Gamma: y = x - s, s \in \Sigma\}.
\]

**Theorem 3.3.** Let \( \mathcal{I} \) be a translation-invariant ideal of subsets of \( \Gamma \), \( \Omega \) a subset of \( \Gamma \) with \( \Gamma \setminus \Omega \in \mathcal{I} \), such that for each \( x \in \Gamma \) the set

\[
\Omega(x) = (x - \Omega) \cap \Omega \cap \bigcap_{geG\setminus\{e\}} (\Omega - gx)_{g}
\]

is non-void. For every \( \phi \in \Phi \), let \( \mathcal{I}_\phi \) be a translation-invariant ideal of subsets of \( \Gamma \). Let \( f \) be a function from \( \Gamma \) into \( A \) such that, given \( \phi \in \Phi \),

(i) the function \( x \rightarrow \langle \phi, f(x) \rangle \) is in \( \ell^\infty(\Gamma) \);

(ii) \( \|\langle \phi, f(\cdot) \rangle\|_{\mathcal{I}_\phi} = 0 \) implies \( \langle \phi, f(x) \rangle = 0 \) for all \( x \in \Omega \);

(iii) (3.1) holds for every \( y \in \Omega \).

Then there exists a unique bounded function \( F \) from \( \Gamma \) into \( (A, \sigma_\phi) \) such that

(2.2) holds for all \( x, y \in \Gamma \) and \( F|_{A} = f|_{A} \).

**Proof.** Let \( F \) be the function from \( \Gamma \) into \( s(\Phi) \) associated with \( f \) by the application of Theorem 3.1. In view of (ii), \( F \) coincides with \( f \) on \( \Omega \). Thus all that we need to prove is that \( F \) is \( A \)-valued.

Given \( x \in \Gamma \), choose \( s \) in \( \Omega(x) \), and let \( t = x - s \). We have \( s \in \Omega \), \( t \in \Omega \), \( s + gt \in \Omega \) for all \( g \in G \setminus \{e\} \), whence \( F(s) = f(s) \), \( F(t) = f(t) \), and \( F(s + gt) = f(s + gt) \) for all \( g \in G \setminus \{e\} \). Consequently,

\[
F(x) = F(s + t) = F(s)F(t) - \sum_{g \in G \setminus \{e\}} F(s + gt)
\]

\[
= f(s)f(t) - \sum_{g \in G \setminus \{e\}} f(s + gt),
\]

which yields the desired conclusion.
THEOREM 3.4. Let \( \mathcal{I} \) be a translation-invariant ideal of subsets of \( \Gamma \), \( \Omega \) a subset of \( \Gamma \) such that \( \Gamma \setminus \Omega \in \mathcal{I} \). For each \( \phi \in \Phi \), let \( \mathcal{I}_\phi \) be a translation-invariant ideal of subsets of \( \Gamma \). Let \( f \) be a function from \( \Gamma \) into a subgroup \( B \) of the multiplicative group of all invertible elements of \( A \) such that, given \( \phi \in \Phi \),

(i) the function \( x \to \langle \phi, f(x) \rangle \) is in \( l^\infty_{\mathcal{I}_\phi}(\Gamma) \);

(ii) \( \| \langle \phi, f(\cdot) \rangle \|_{\mathcal{I}_\phi} \neq 0 \);

(iii) for every \( y \in \Omega \)

\[ \langle \phi, f(\cdot + y) \rangle = \mathcal{I}_\phi \langle \phi, f(\cdot) \rangle \langle \phi, f(y) \rangle. \]

Then there exists a unique bounded function \( F \) from \( \Gamma \) into \( (A, \sigma_\phi) \) such that for all \( x, y \in \Gamma \)

\[ F(x + y) = F(x) F(y) \]

and \( F|_\Omega = f|_\Omega \). The range of \( F \) is contained in \( B \).

Proof. Let \( F \) be the function from \( \Gamma \) into \( s(\Phi) \) associated with \( f \) by the application of Theorem 3.1. Clearly, \( F \) coincides with \( f \) on \( \Omega \). We shall show that \( F \) is \( B \)-valued.

Given \( x \in \Gamma \), let \( y \) be an element of \( \Omega \cap (\Omega - x) \). Then \( x + y \in \Omega \), and so

\[ f(x + y) = F(x + y) = F(x) F(y) = F(x) f(y), \]

whence

\[ F(x) = f(x + y) f(y)^{-1}. \]

The proof is complete.

The following is a generalization of de Bruijn's theorem mentioned in the Introduction.

THEOREM 3.5. Let \( \mathcal{I} \) be a translation-invariant ideal of subsets of \( \Gamma \), \( \Omega \) a subset of \( \Gamma \) such that \( \Gamma \setminus \Omega \in \mathcal{I} \), and \( H \) a locally compact Abelian group. For each \( \chi \in \hat{H} \) let \( \mathcal{I}_\chi \) be a translation-invariant ideal of subsets of \( \Gamma \). Let \( f \) be a function from \( \Gamma \) into \( H \) such that for every \( \chi \in \hat{H} \) and every \( y \in \Omega \)

\[ (f(\cdot + y), \chi) = \mathcal{I}_\chi (f(\cdot), \chi)(f(y), \chi). \]

Then there exists a unique function \( F \) from \( \Gamma \) into \( H \) such that for all \( x, y \in \Gamma \)

\[ F(x + y) = F(x) + F(y) \]

and \( F|_\Omega = f|_\Omega \).

Proof. The theorem follows immediately from the foregoing one upon identifying \( H \) with the group of all continuous characters of \( \hat{H} \), which is a subgroup of the group of all invertible elements of the Banach algebra \( l^\infty(\hat{H}) \), and taking \( \Phi \) to be the collection of all evaluation functionals on \( l^\infty(\hat{H}) \) corresponding to points in \( \hat{H} \).
REFERENCES


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