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On Banach algebras which are Hilbert spaces

Let $F$ denote a field that is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Let $H$ denote the real quaternion algebra.

An algebra $A$ over $F$ with identity $e$ is called a **Hilbert algebra with identity** if $A$ is a Hilbert space such that the norm derived from an inner product has the following properties:

1° $||e|| = 1,$
2° $||a \cdot b|| \leq ||a|| \cdot ||b||$ for $a, b \in A$.

If $F = \mathbb{R}$, $A$ is called a **real Hilbert algebra**, and if $F = \mathbb{C}$, a complex **Hilbert algebra**.

L. Ingelstam proved in [2] the following

**Theorem.** A real Hilbert algebra with identity is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $H$.

In the present note we shall give a new simpler proof of this theorem. We shall begin our considerations with the following

**Lemma.** Let $A$ be a real Hilbert algebra with identity $e$. Let $\Phi$ be a multiplicative linear functional on $A$. Then $\Phi$ is injective.

**Proof.** In virtue of the Riesz theorem, $\Phi$ represents in the form

$$\Phi(a) = (e, a),$$

where $e, a \in A$. It is well known that $||\Phi|| = 1$ for an arbitrary multiplicative linear functional $\Phi$ on a Banach algebra with identity. Hence and by (1) we obtain

$$||a|| = 1.$$

By the Cauchy–Schwarz inequality and by 1°, (2) we have

$$1 = \Phi(e) = (e, a) \leq ||e|| \cdot ||a|| = 1.$$

In particular $(e, a) = ||e|| \cdot ||a||$. But the equality in the Cauchy–Schwarz inequality occurs exactly in the case when both $a$ and $e$ are linearly de-
pendent. Therefore we can write \( a = \lambda e, \lambda \in \mathbb{R} \). Using \( 1^o, (3) \) we obtain
\[
\lambda = \lambda(e, e) = (a, e) = 1,
\]
so actually we have \( a = e \) and
\[
(4) \quad \Phi(x) = (x, e).
\]

Now let us consider the following linear operator
\[
T_a(x) = a \cdot x, \quad a, x \in A.
\]
Using \( 2^o \) and the fact that \( T_a(e) = a \) we obtain
\[
(5) \quad \|T_a\| = \|a\|.
\]
Let \( T^*_a \) denote the adjoint operator. By the elementary fact that \( \|T^*_a\| = \|T_a\| \) and by \( (5) \) we obtain
\[
(6) \quad \|T^*_a\| = \|a\|.
\]
Now by the Cauchy–Schwarz inequality and by \( 1^o, (6) \) we have
\[
(7) \quad \|a||^2 = (T_a(e), a) = (e, T^*_a(a)) \leq \|e\| \cdot \|T^*_a(a)\| \leq \|T^*_a\| \cdot \|a\| = \|a\|^2.
\]
Hence, as before, \( T^*_a(a) = \lambda e \) for some \( \lambda \in \mathbb{R} \), and by \( 1^o, (7) \) we have
\[
\lambda = \lambda(e, e) = (e, T^*_a(a)) = \|a||^2.
\]
Consequently
\[
(8) \quad T^*_a(a) = \|a||^2 \cdot e.
\]
Now by \( (8) \) we obtain for \( a, b \in A \)
\[
(9) \quad (a \cdot b, a) = (T_a(b), a) = (b, T^*_a(a)) = \|a||^2 \cdot (b, e).
\]
By \( (9) \), \( 1^o \) we have for \( a \in A \)
\[
(10) \quad ((a + e) \cdot a, a + e) - (a^2, a) = \|a + e||^2 \cdot (a, e) - \|a||^2 \cdot (a, e)\]
\[
= (1 + 2(a, e)) \cdot (a, e).
\]
But
\[
(11) \quad ((a + e) \cdot a, a + e) - (a^2, a) = \|a||^2 + (a^2, e) + (a, e).
\]
By \( (4) \) we have
\[
(12) \quad (a^2, e) = \Phi(a^2) = (\Phi(a))^2 = (a, e)^2.
\]
Taking account of \( (10), (11), (12) \) we obtain \( \|a|| = |(a, e)| = |\Phi(a)| \). Hence \( \ker \Phi = \{0\} \). The proof of the lemma is completed.

Now we shall prove the main theorem. Let \( A \) be a real Hilbert algebra with identity. In order to prove that \( A \) is isomorphic to \( \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \) it is sufficient to show that \( A \) is a real division algebra (cf. Theorem 7,
Chapter I, § 14 of F. F. Bonsal, J. Duncan [1]). Let $x$ be an arbitrary non-zero element of $A$. Let us consider the smallest closed subalgebra $A_0$ with identity containing $x$. $A_0$ is a commutative real Hilbert algebra with identity. Suppose that $x$ has no inverse in $A_0$. Then $x$ belongs to a certain maximal ideal $M$. Obviously $M$ is closed. Let $\varphi$ be the canonical homomorphism of $A_0$ onto $A_0/M$. $A_0/M$ is a commutative real division algebra and therefore $A_0/M$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. In both cases $A_0/M$ contains a closed subalgebra $B$ isomorphic to $\mathbb{R}$. Let $\eta$ be an isomorphism of $B$ onto $\mathbb{R}$. Since $\varphi$ is continuous, it follows that $\varphi^{-1}(B)$ is a real Hilbert algebra with identity. $\Phi = \eta \circ \varphi$ is a multiplicative linear functional on $\varphi^{-1}(B)$. In virtue of lemma, $\Phi$ is injective. In particular, $M = \{0\}$ contrary to our assumption of non-triviality of $M$. Consequently $x$ has inverse in $A_0$, and so in $A$. $A$ is then a real division algebra. This completes the proof.

An immediate consequence of our theorem is the following

**Corollary.** A complex Hilbert algebra with identity is isomorphic to $\mathbb{C}$.

**Proof.** Consider a complex Hilbert algebra $A$ with identity as a real Hilbert algebra with the inner product $\langle x, y \rangle = \text{Re}(x, y)$ for $x, y \in A$. In virtue of our theorem, $A$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Among the last three algebras $\mathbb{C}$ is the only complex Banach algebra. This remark completes the proof.

**References**
