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ON SOME FUNCTIONALS ON BERGMAN SPACES

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SUMMARY. We prove that if Ω is an open subset of $\boldsymbol{C}$ with nontrivial Bergman space, then the functionals $f \mapsto f^{(k)}(w)$ $(w \in \Omega,\ k \in \boldsymbol{N} \cup \{0\})$ on the Bergman space of Ω are nonzero and linearly independent.

Let Ω be an open subset of $\boldsymbol{C}$. Let $L^2H(\Omega)$ be the Bergman space of Ω, that is, the space of all holomorphic Lebesgue square integrable functions on Ω. Suppose that $L^2H(\Omega) \neq \{0\}$. For each $w \in \Omega$ and each $n \in \boldsymbol{N} \cup \{0\}$, let $\delta_w^{(n)}$ denote the linear continuous functional on $L^2H(\Omega)$ defined by

$$\delta_w^{(n)}(f) = f^{(n)}(w) \quad \big(f \in L^2H(\Omega)\big).$$

M. Skwarczyński [3] showed that the functionals $\delta_w^{(0)}$ $(w \in \Omega)$ are nonzero and linearly independent. Refining an argument of Skwarczyński, we prove the following generalization of that result.

Theorem. *The functionals* $\delta_w^{(n)}$ $(w \in \Omega,\ n \in \boldsymbol{N} \cup \{0\})$ *are nonzero and linearly independent.*

Proof. Suppose that there exist $a_i \in \boldsymbol{C} \setminus \{0\}$, $w_i \in \Omega$, and $n_i \in \boldsymbol{N} \cup \{0\}$ $(1 \leqslant i \leqslant N)$ such that $(n_i, w_i) \neq (n_j, w_j)$ whenever $i \neq j$ $(1 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant N)$ and

$$\sum_{i=1}^{N} a_i \delta_{w_i}^{(n_i)} = 0. \tag{1}$$

Let $I_1 = \{i \in \{1, \ldots, N\} \colon w_i = w_1\}$ and $I_2 = \{1, \ldots, N\} \setminus I_1$. Let $m = \max\{n_i \colon i \in I_1\}$ and let $j \in I_1$ satisfy $n_j = m$. By a theorem of Skwarczyński–Wiegerinck [3, 4], being nontrivial, the space $L^2H(\Omega)$ is in fact infinite-dimensional. Hence the space

$$H = \bigcap_{0 \leqslant k \leqslant m-1} \ker \delta_{w_1}^{(k)} \cap \bigcap_{\substack{0 \leqslant k \leqslant n_i \\ i \in I_2}} \ker \delta_{w_i}^{(k)}$$

is nontrivial. Let f be any nonzero element of H. Let

$$r = \min\{k \in \boldsymbol{N} \cup \{0\} \colon f^{(m+k)}(w_1) \neq 0\}.$$

For each $z \in \Omega \setminus \{w_1\}$, set $g(z) = (z - w_1)^{-r} f(z)$. Since all the derivatives of f at w_1 of order no greater than $m + r - 1$ vanish, g can be extended in a unique way to a holomorphic function on Ω. Choose $\delta > 0$ so that the disc $D(w_1, \delta) = \{z \in \boldsymbol{C} \colon |z - w_1| < \delta\}$ is contained in Ω. Then

$$\int_\Omega |g|^2 \, d\lambda \leqslant \max\{|g(z)|^2 \colon z \in D(w_1, \delta)\}\, \lambda\big(D(w_1, \delta)\big) + \delta^{-2r} \int_{\Omega \setminus D(w_1, \delta)} |f|^2 \, d\lambda,$$

which shows that g is in $L^2H(\Omega)$; here λ stands for the Lebesgue measure on Ω. Now an easy computation establishes that $g \in H$ and that

$$g^{(m)}(w_1) = \frac{m!}{(m+r)!} f^{m+r}(w_1) \neq 0. \tag{2}$$

Hence

$$\sum_{i=1}^{N} a_i \delta_{w_i}^{(n_i)}(g) = a_j g^{(m)}(w_1).$$

The latter identity jointly with (1) and (2) implies that $a_j = 0$, a contradiction. The proof is complete.

We conclude with two remarks. First, Example II.1 in [2] and the examples of finite-dimensional Bergman spaces in [3, 4] show that the natural generalization of the theorem established to open subsets of $\boldsymbol{C}^n$ $(n \geqslant 2)$ is in general not valid. Second, it follows from the above theorem that any domain in $\boldsymbol{C}$ with nontrivial Bergman space has property $\mathscr{A}$ as introduced in [1]. Therefore Theorem 3.1 of [1] is subsumed in Theorem 3.2 of that same paper.

References

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В. Хойнацки, **О некоторых функционалах на простанствах Бергмана**

В работе доказывается, что если Ω — открытое подмножество с нетривиальным простанством Бергмана, то функционалы $f \mapsto f^{(k)}(w)$ $(w \in \Omega,\ k \in \boldsymbol{N} \cup \{0\})$ на простанстве Бергмана множества Ω ненулевы и линейно независимы.

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