Topological containment in graphs: New characterizations from computer search

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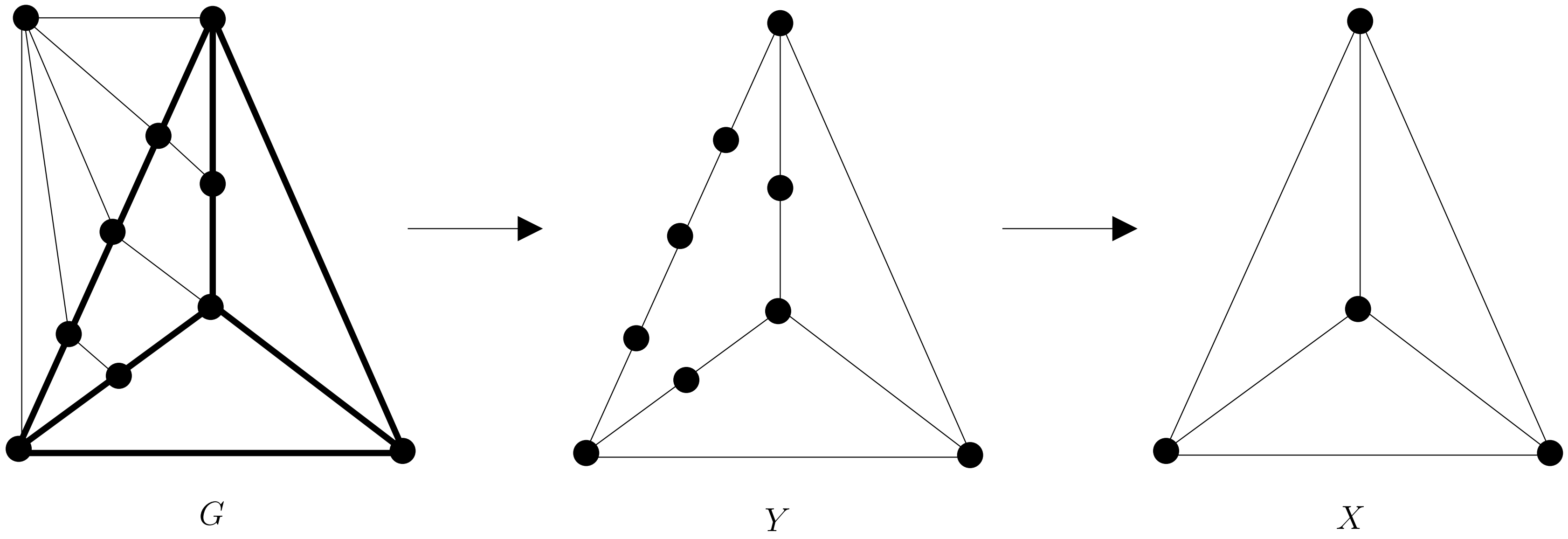
1 Topological containment

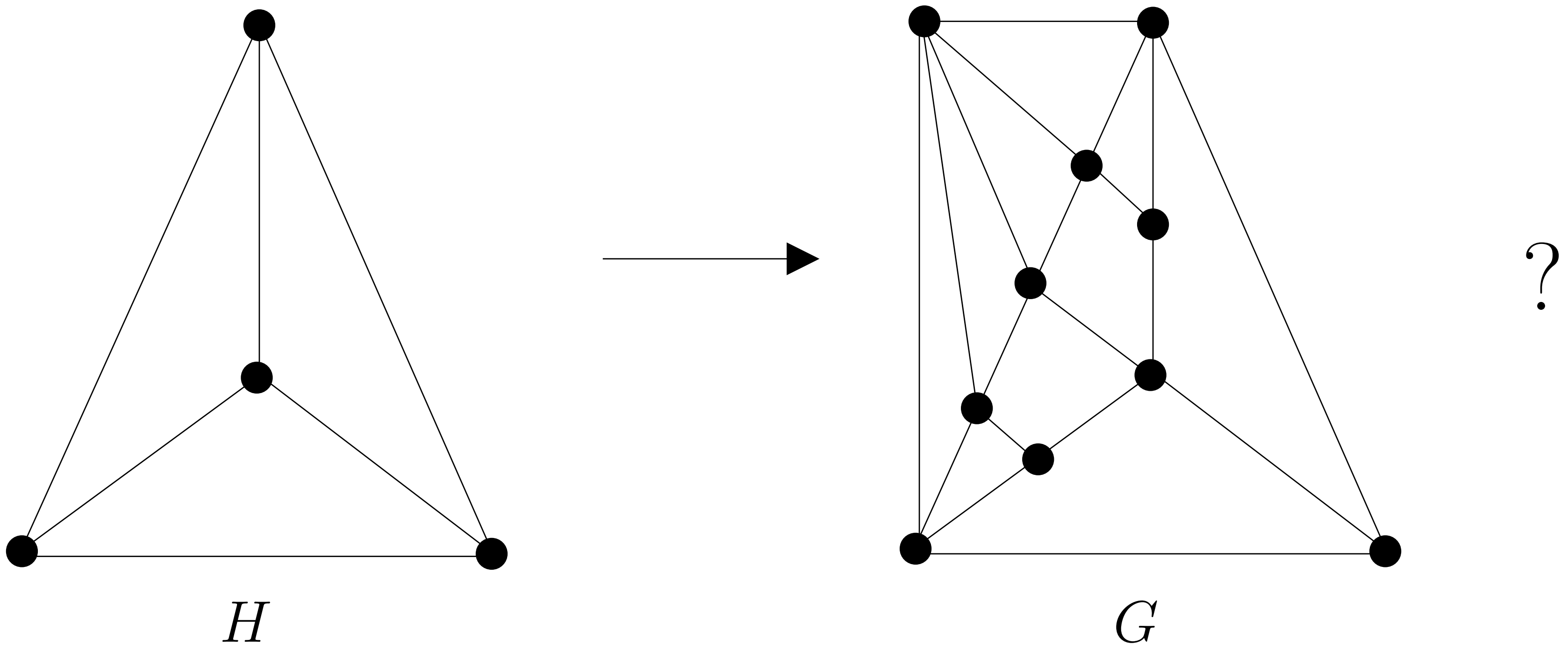


- Formally: G topologically contains X *iff G contains some subgraph Y such that X can be obtained from Y by performing a series of contractions limited to edges that have at least one endvertex of degree 2.*

- Also: Y is an X-*subdivision*; G *contains* an X-subdivision

Problem of topological containment:

- $TC(H)$: For some fixed *pattern graph* H — given a graph G, does G contain an H-subdivision?



2 Robertson and Seymour results

DISJOINT PATHS (DP)

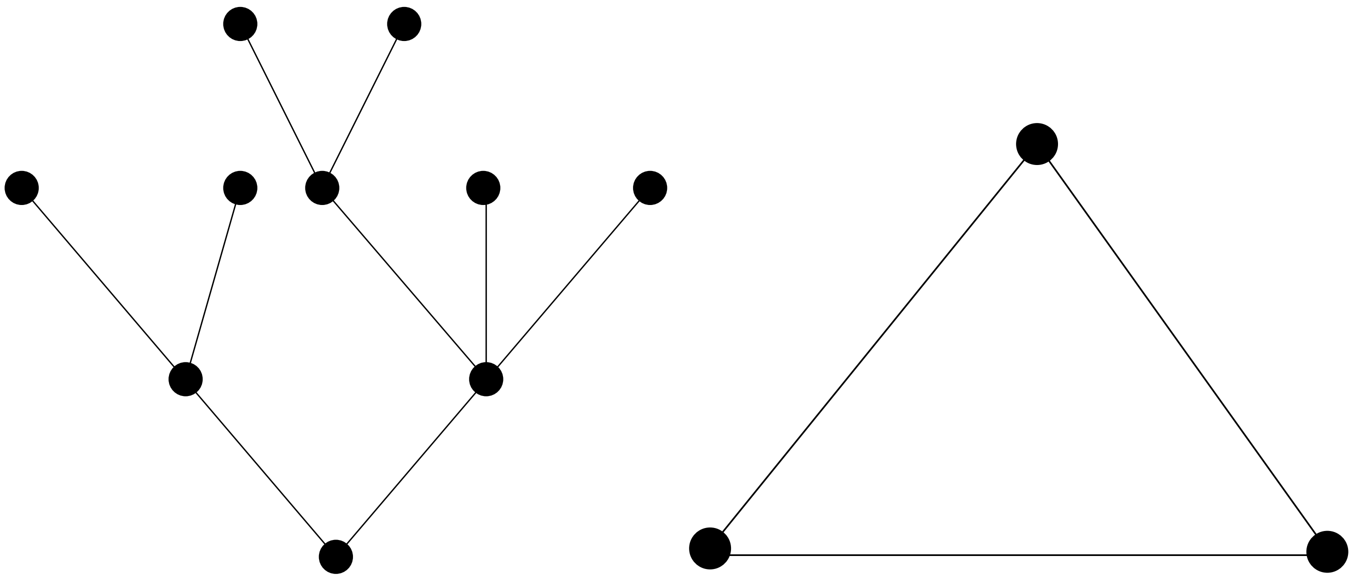
Input: Graph G; pairs $(s_1, t_1), ..., (s_k, t_k)$ of vertices of G.

Question: Do there exist paths $P_1, ..., P_k$ of G, mutually vertex-disjoint, such that P_i joins s_i and t_i $(1 \leq i \leq k)$?

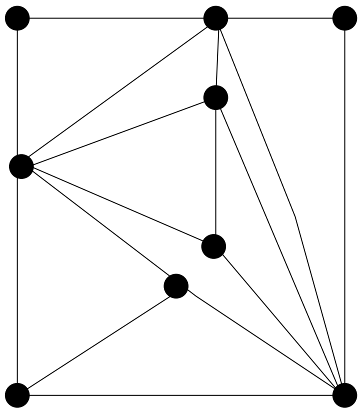
- DISJOINT PATHS is in P for any fixed k.
- This implies topological containment problem for fixed H is also in P — use DP repeatedly.
- We know p-time algorithms must exist for topological containment, but practical algorithms not given — huge constants.
- Finding efficient, practical algorithms for particular pattern graphs is still an important task.

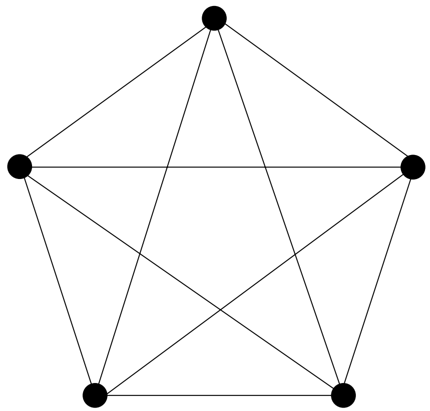
3 Examples of good characterizations

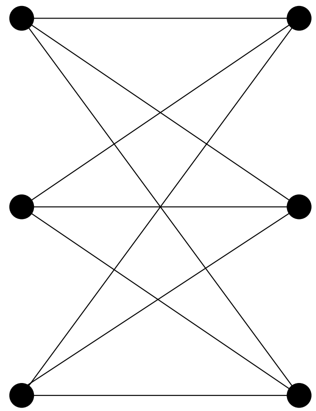
- Trees — K_3



- Kuratowski (1930) — K_5 or $K_{3,3}$ in non-planar graphs

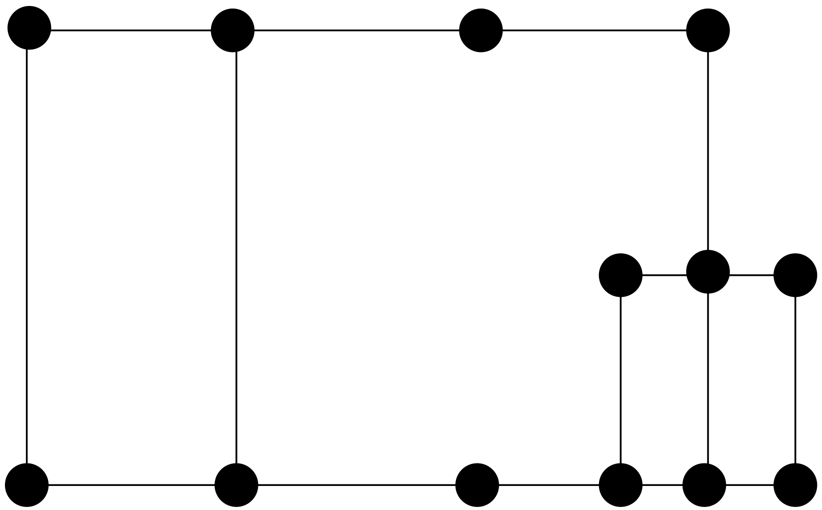


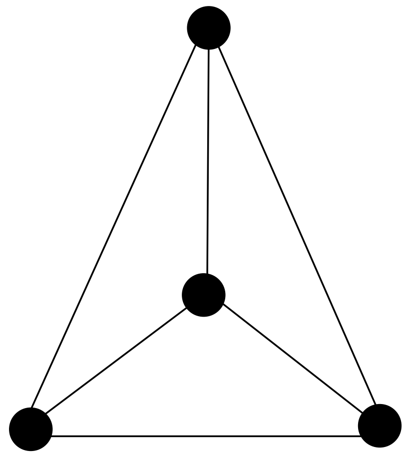




- Wagner (1937) and Hall (1943) strengthened this result to characterize graphs with no $K_{3,3}$-subdivisions

- Duffin (1965) — K_4 in non-series-parallel graphs

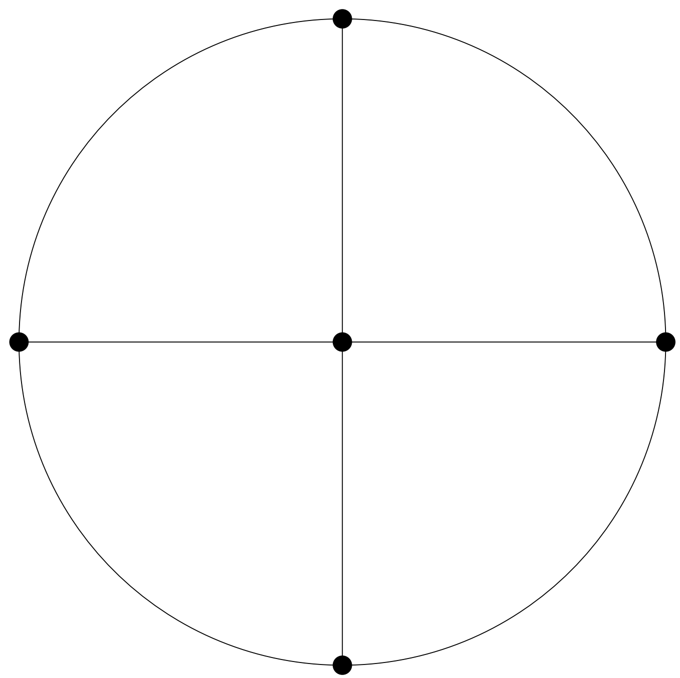




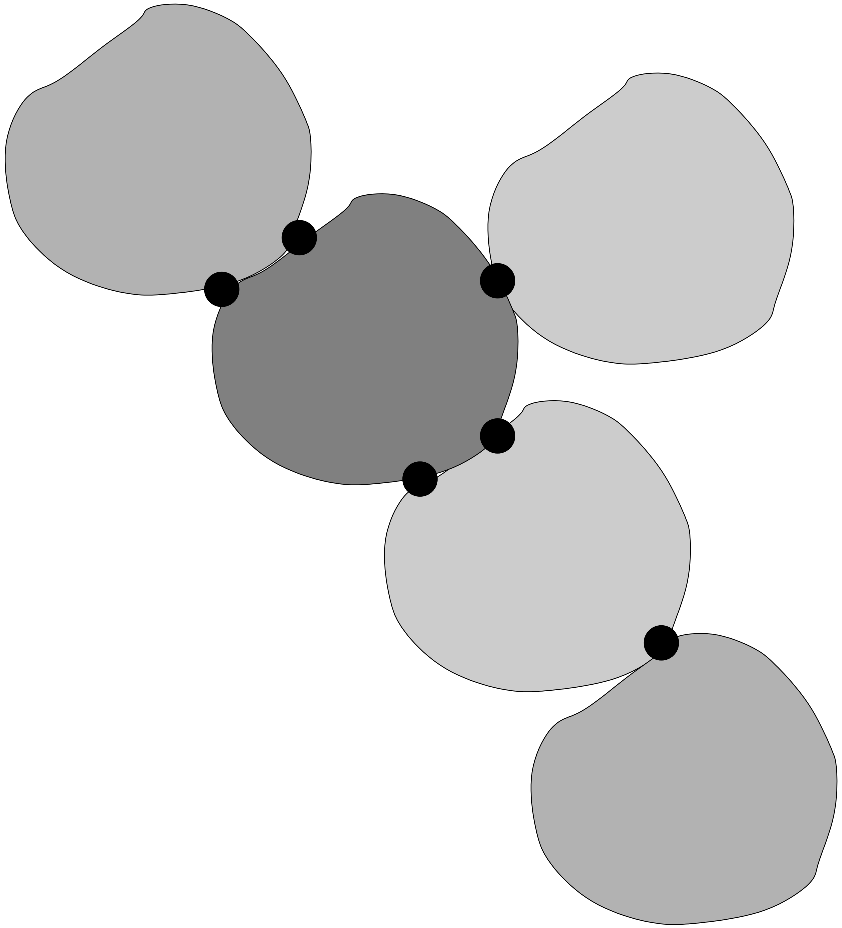
4 Previous results on wheel subdivisions

4.1 W_4 — wheel with four spokes

- Farr (1988) — If G is 3-connected, G contains a W_4-subdivision if and only if G has a vertex of degree ≥ 4



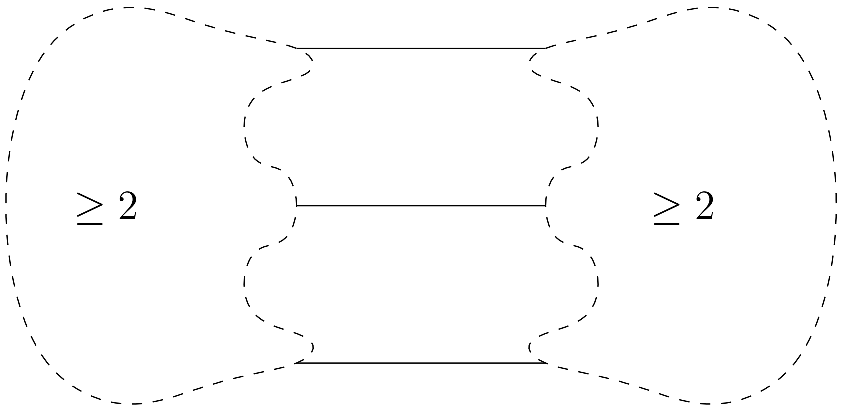
Structure of graphs with no W_4-subdivisions



4.2 W_5 — wheel with five spokes

Theorem (Farr, 88).

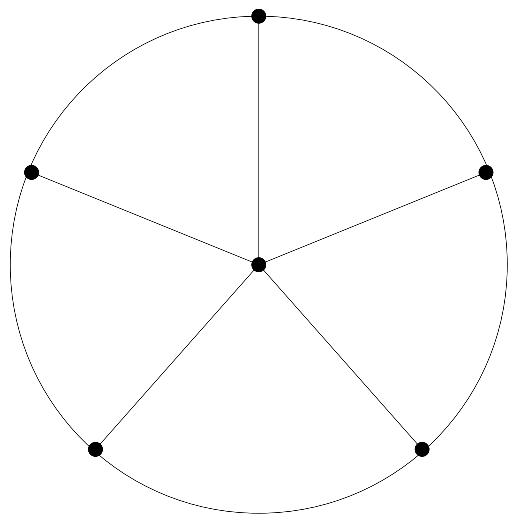
*Let G be 3-connected, with no **internal 3-edge-cutset** . . .*



Internal 3-edge-cutset

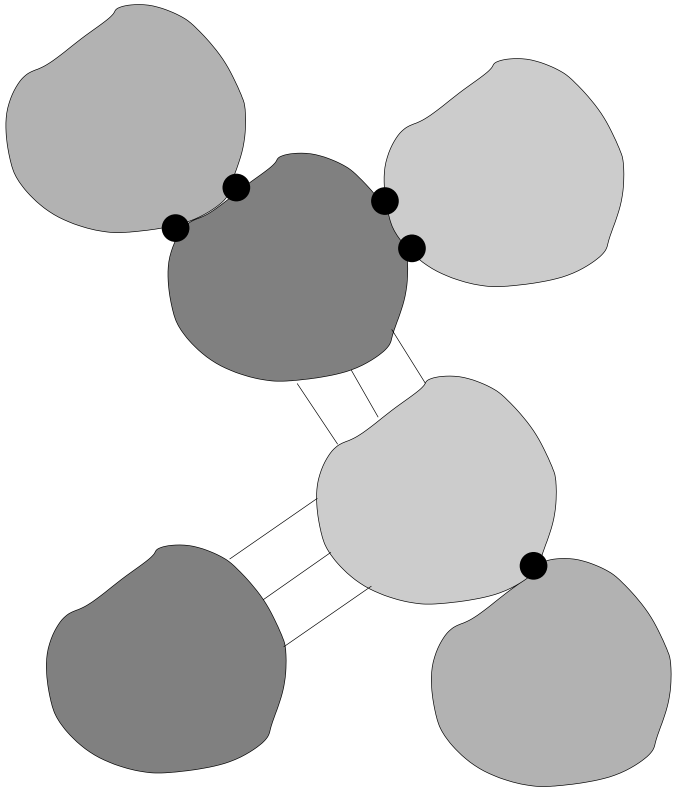
Theorem (Farr, 88).

Let G be 3-connected, with no internal 3-edge-cutset. Then G has a W_5-subdivision if and only if G has a vertex v of degree at least 5 and a circuit of size at least 5 which does not contain v.



W_5: wheel with five spokes

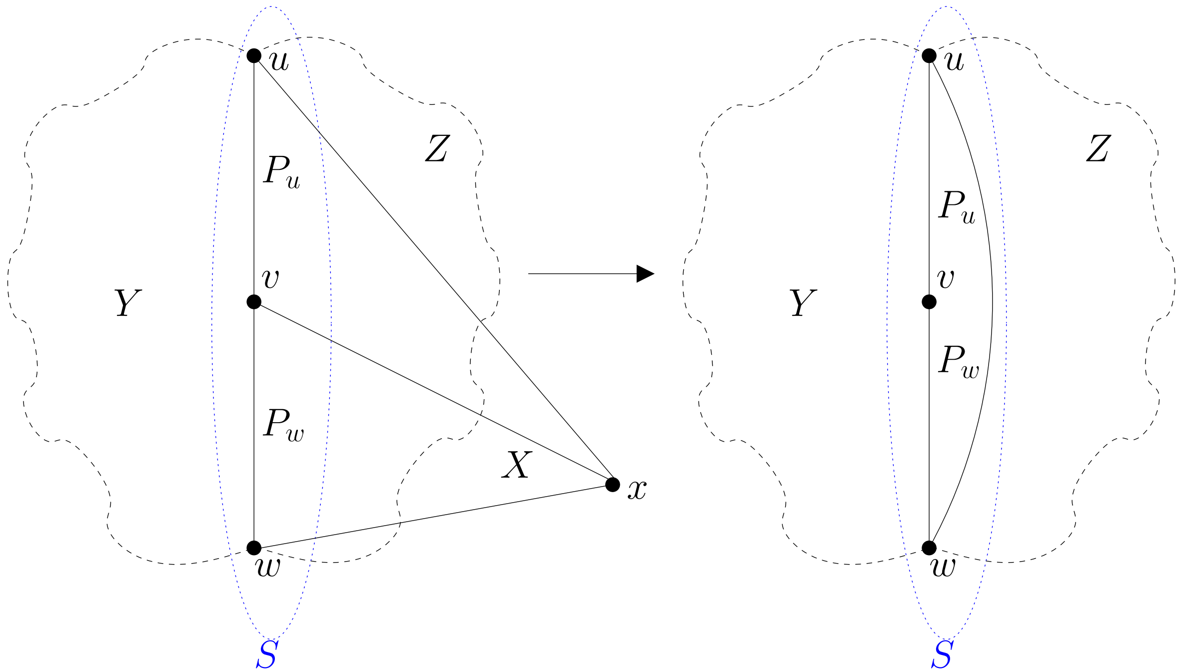
Structure of graphs with no W_5-subdivisions



5 Strengthened W_5 result

- Uses a forbidden reduction

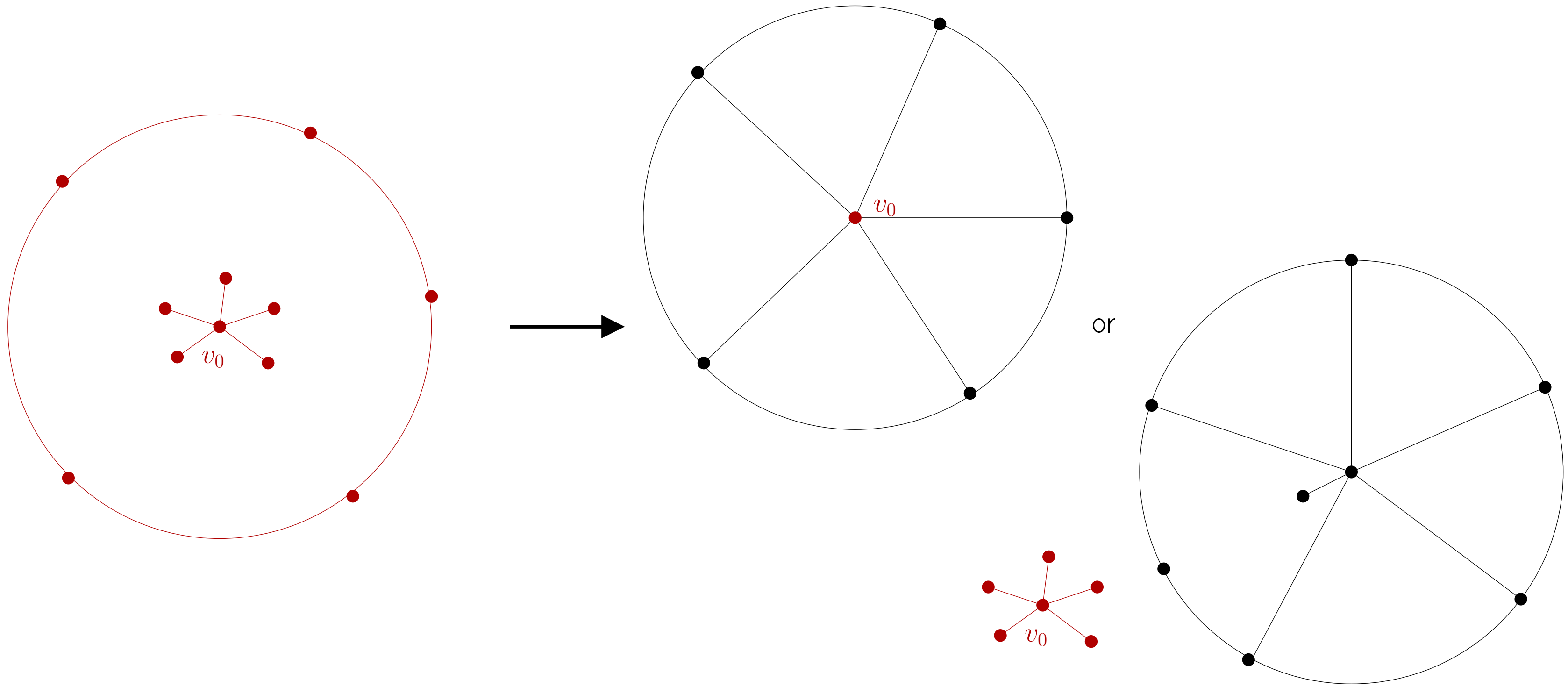
Reduction 1A



- X, Y, and Z are *bridges* of $G|S$.

Theorem.

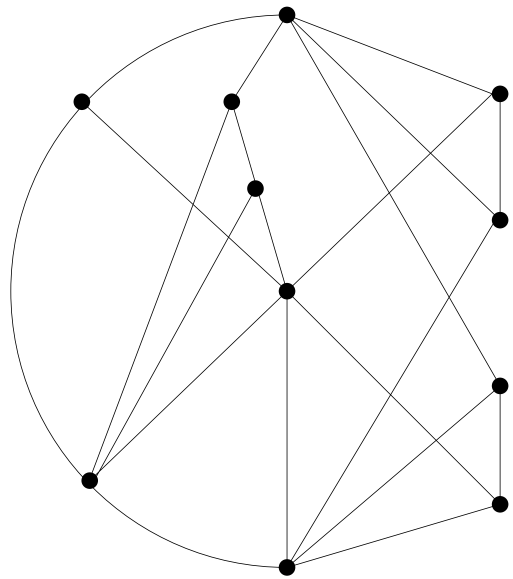
Let G be a 3-connected graph, with no internal 3-edge-cutset, such that Reduction 1A cannot be performed on G. Let v_0 be a vertex of degree ≥ 5 in G. Suppose there is a cycle of size at least 5 in G which does not contain v_0. Then either G has a W_5-subdivision centred on v_0, or G has a W_5-subdivision centred on some vertex v_1 of degree ≥ 6, with a rim of size at least 6.



6 Characterization of graphs with no W_6-subdivision

Theorem.

*Let G be a 3-connected graph that is not topologically contained in the **graph A** . . .*

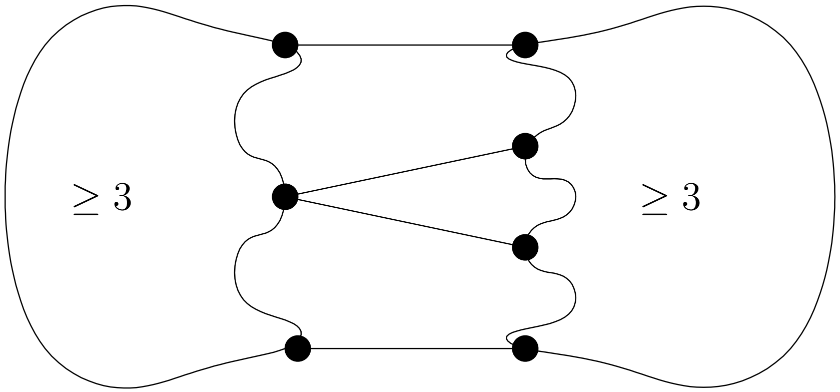


Graph A

Theorem.

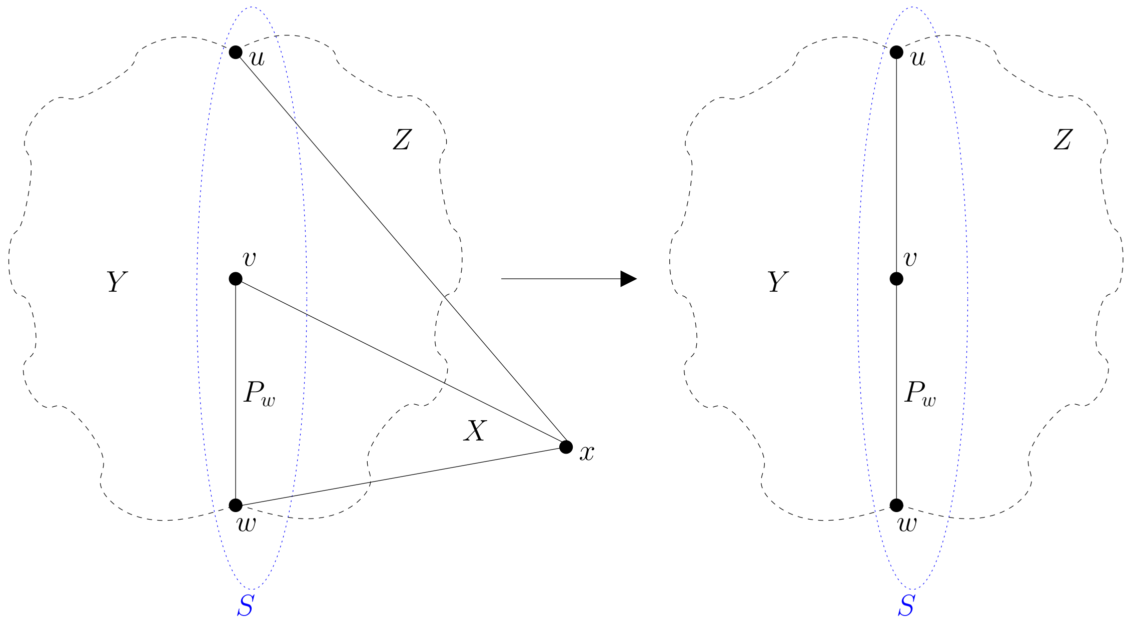
Let G be a 3-connected graph that is not topologically contained in the graph A.

Suppose G has no internal 3-edge-cutsets, no ***internal 4-edge-cutsets*** *. . .*



Theorem.

Let G be a 3-connected graph that is not topologically contained in the graph A. Suppose G has no internal 3-edge-cutsets, no internal 4-edge-cutsets, and is a graph on which neither Reduction 1A nor ***Reduction 2A*** *can be performed, for* $k = 6\ldots$

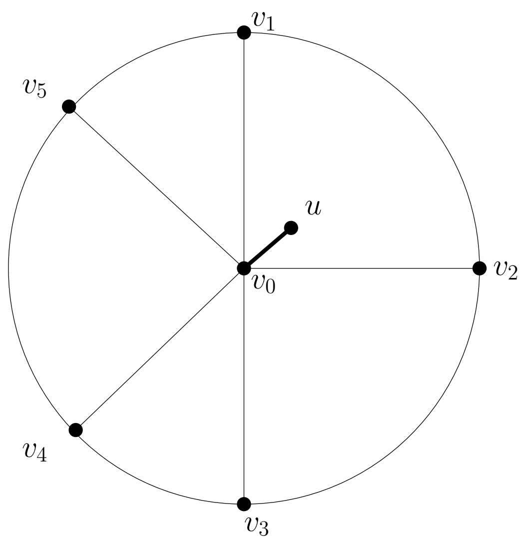


Theorem.

Let G be a 3-connected graph that is not topologically contained in the graph A. Suppose G has no internal 3-edge-cutsets, no internal 4-edge-cutsets, and is a graph on which neither Reduction 1A nor Reduction 2A can be performed, for $k = 6$. Then G has a W_6-subdivision if and only if G has a vertex v of degree at least 6.

Proof — a summary.

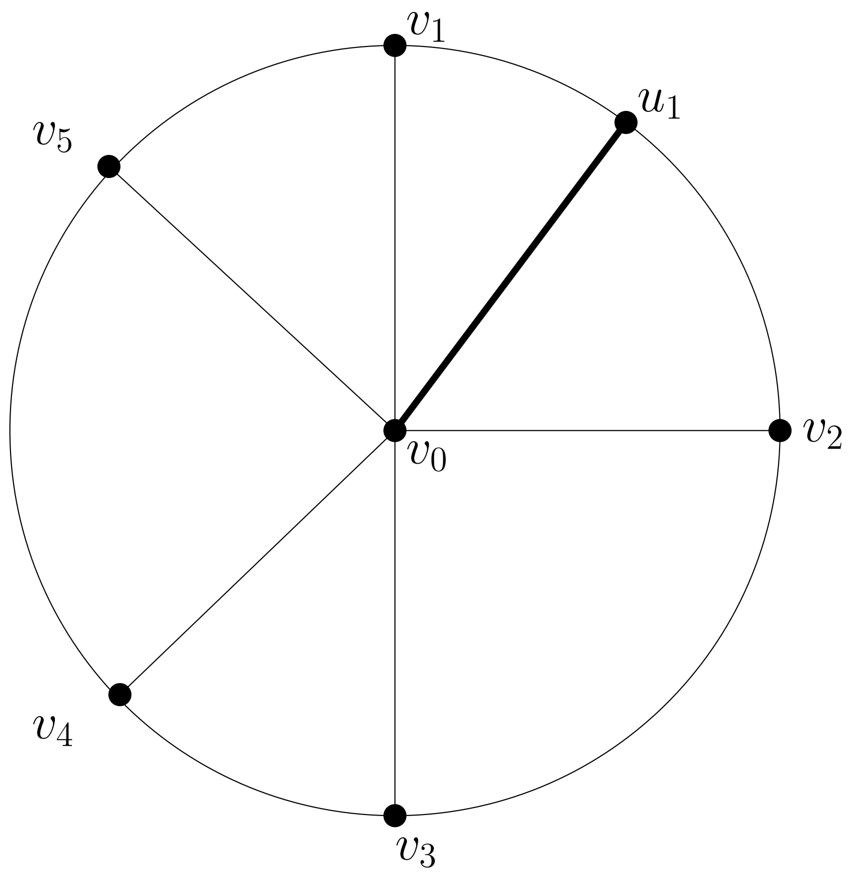
- Suppose the conditions of the hypothesis hold for some graph G.
- By the strengthened W_5 result above, there exists some vertex v_0 of degree ≥ 6 in G that has a W_5-subdivision H centred on it.



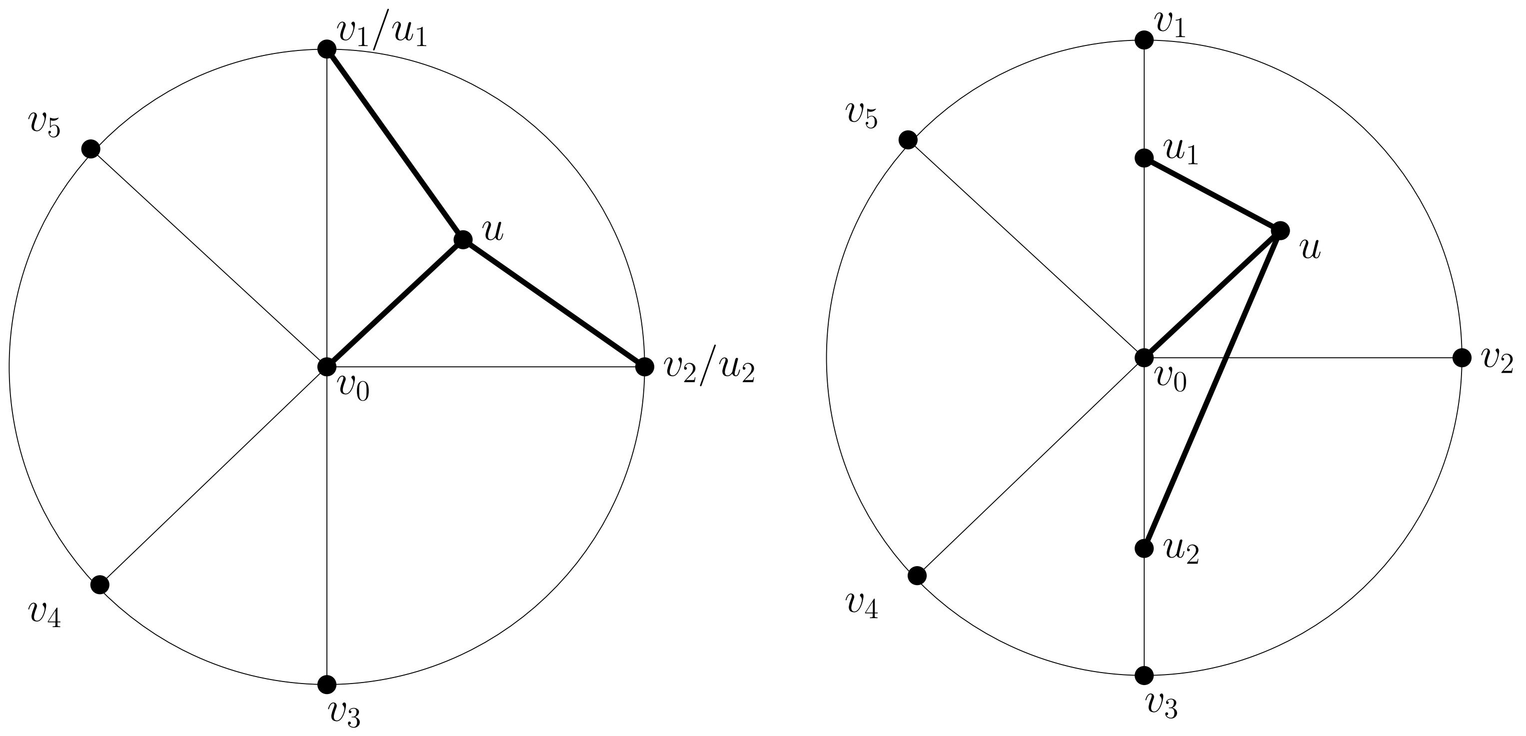
- How does u connect to the rest of H in order to preserve 3-connectivity?

Three possibilities:

(a) Path from v_0 to some vertex u_1 on the rim of the W_5-subdivision, not meeting any spoke.



(b) Two paths from u to two separate spokes of H.



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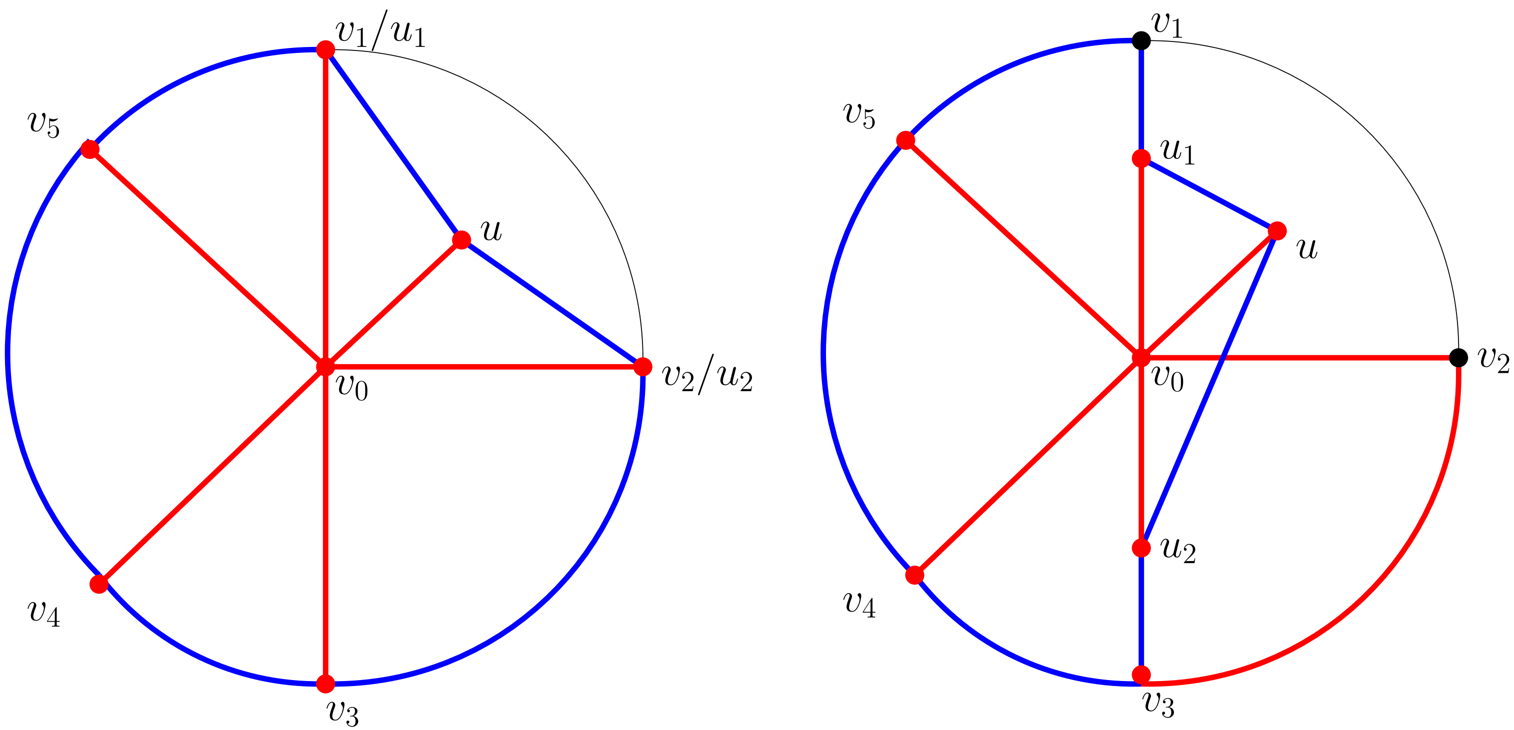
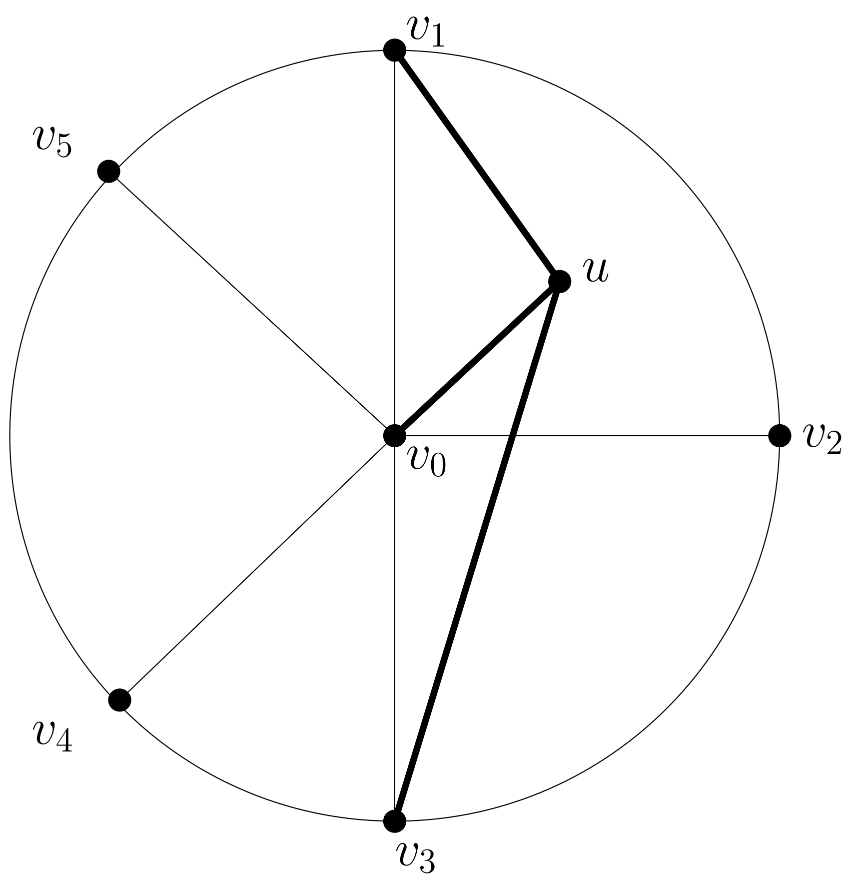


- Dealing with one particular case takes up the majority of the proof:

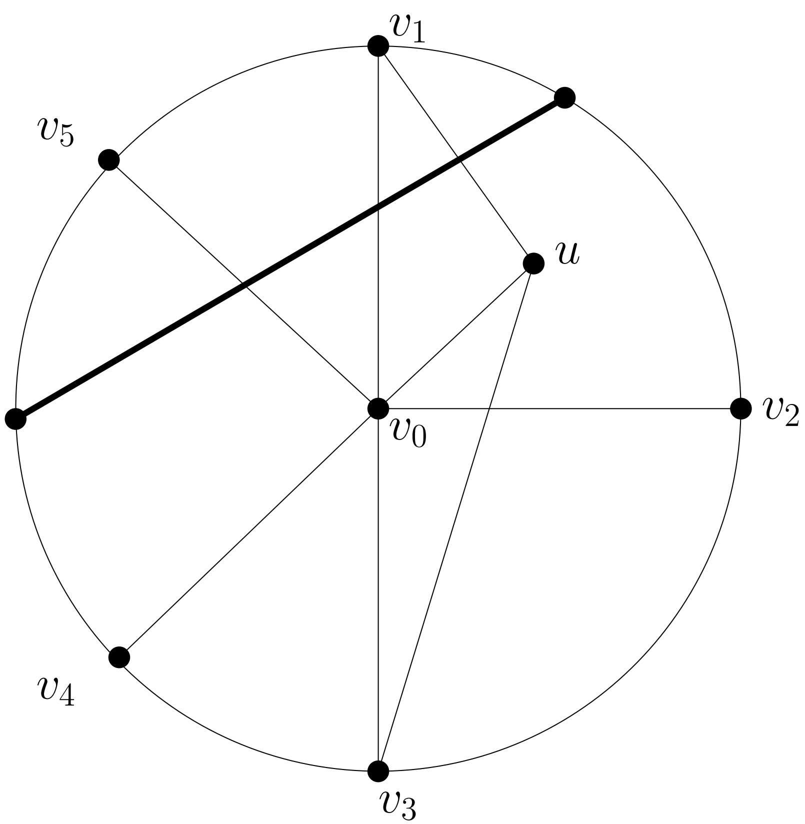


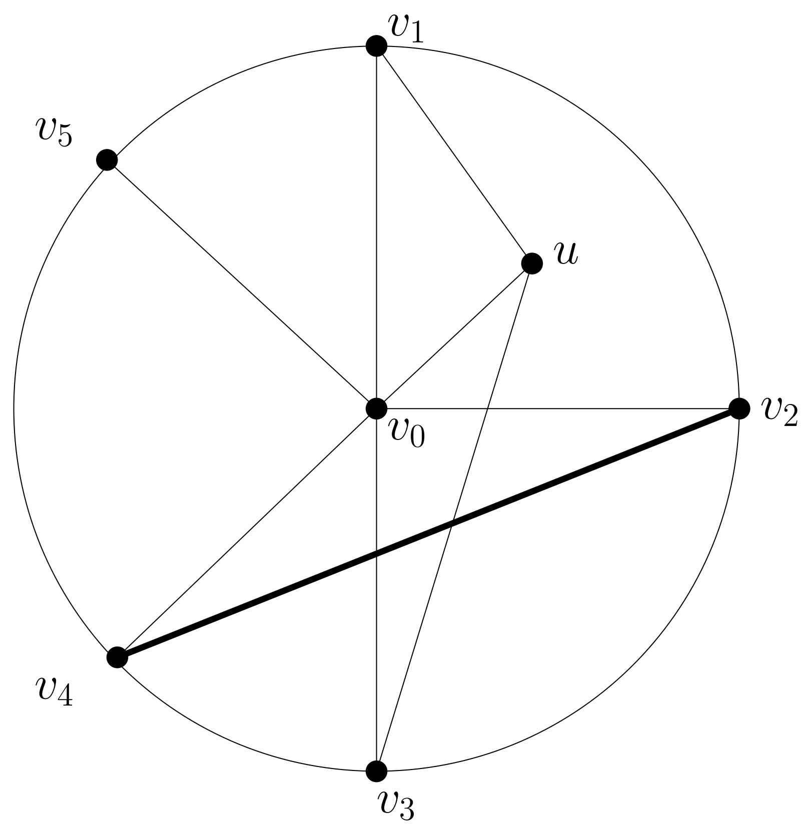
- This graph meets 3-connectivity requirements, but Reduction 1A can be performed on it.
- So there must be more structure to the graph.
- More in-depth case analysis required, based on different ways of adding this structure.
- Program developed in C to automate parts of this analysis; parts of proof depend on results generated by this program.

The program:

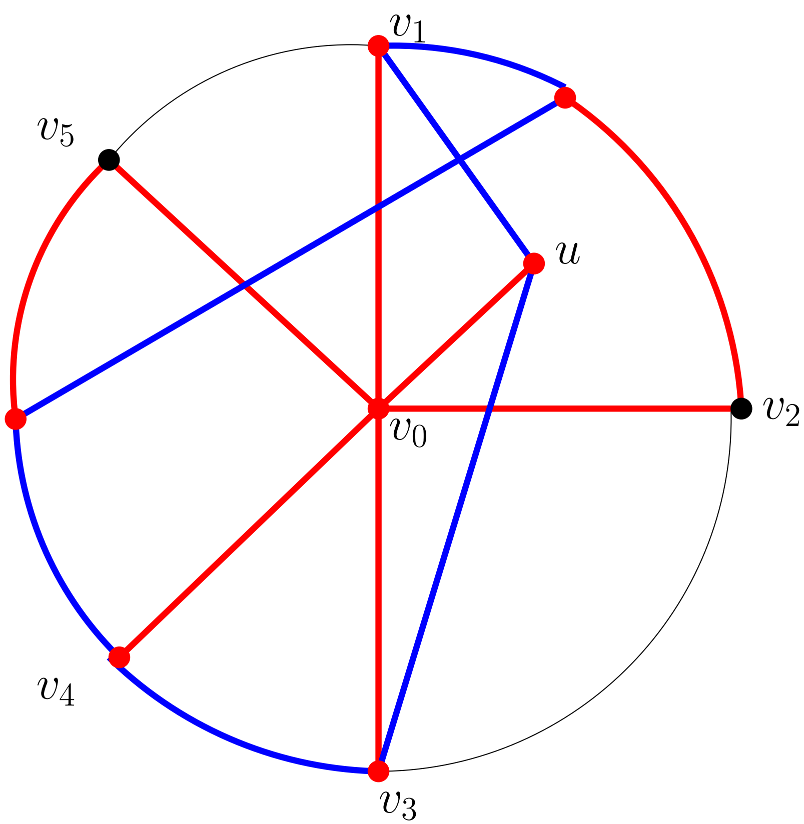
- constructs the various simple graphs that arise as cases in the proof, and
- tests each graph for the presence of a W_6-subdivision.

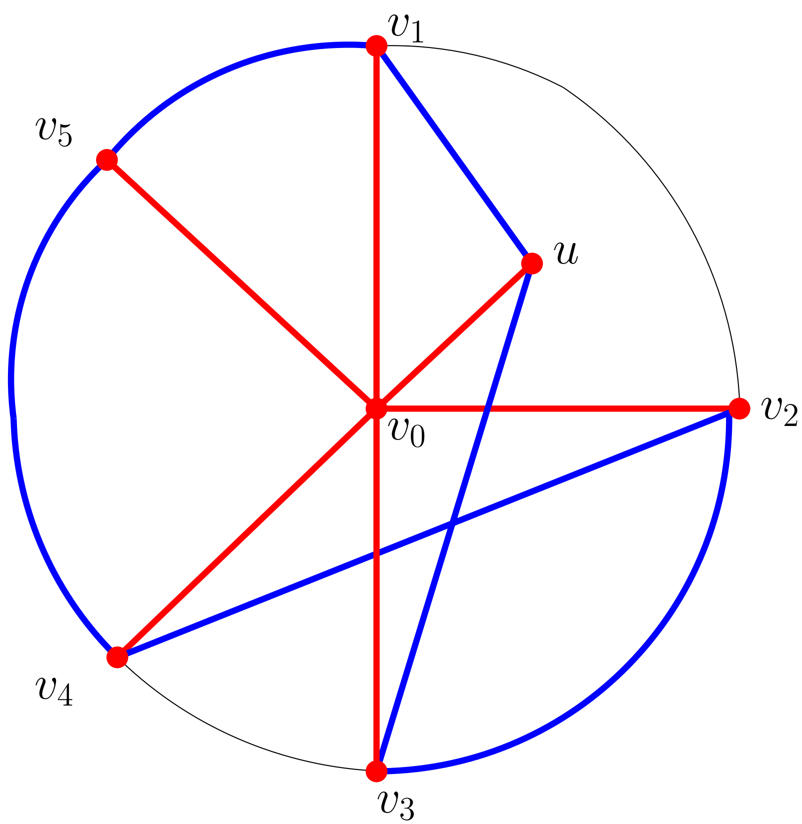
Examples:





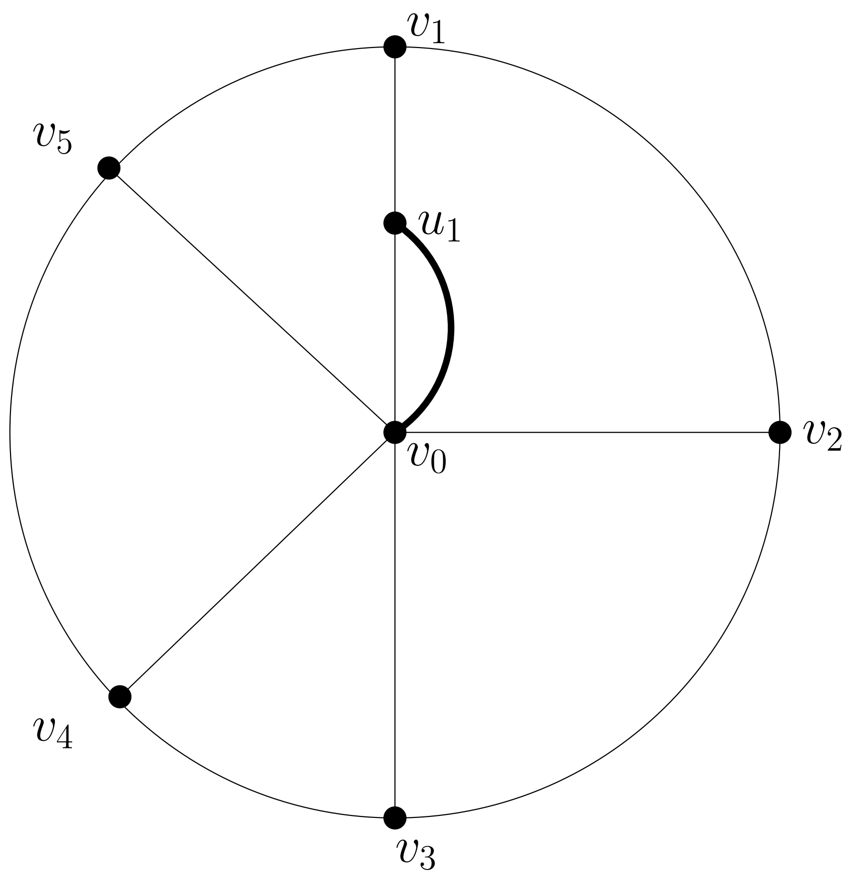
Examples:





- The program determines if a W_6-subdivision is present by recursively testing all subgraphs obtained by removing a single edge from the input graph.
- Base cases are W_6-subdivisions or graphs that have too few vertices or edges to contain such a subdivision.
- Naive algorithm; takes exponential time, but is sufficient for the small input graphs that arise as cases in the proof.
- Once the possibility of performing reductions and the presence of internal 3- and 4-edge-cutsets is eliminated, all resulting graphs are found to either:
  - contain a W_6-subdivision; or
  - be topologically contained in Graph A.

(c) Path from v_0 to some vertex u_1 on one of the spokes of the W_5-subdivision, such that this path that does not meet H except at its end points.



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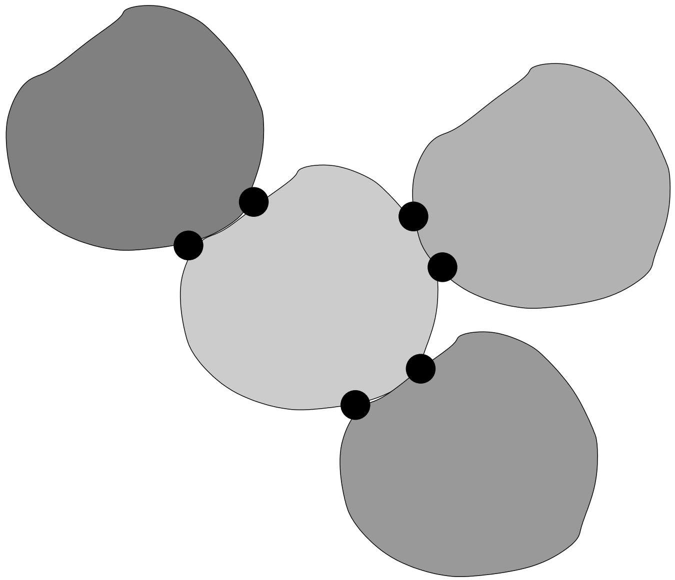
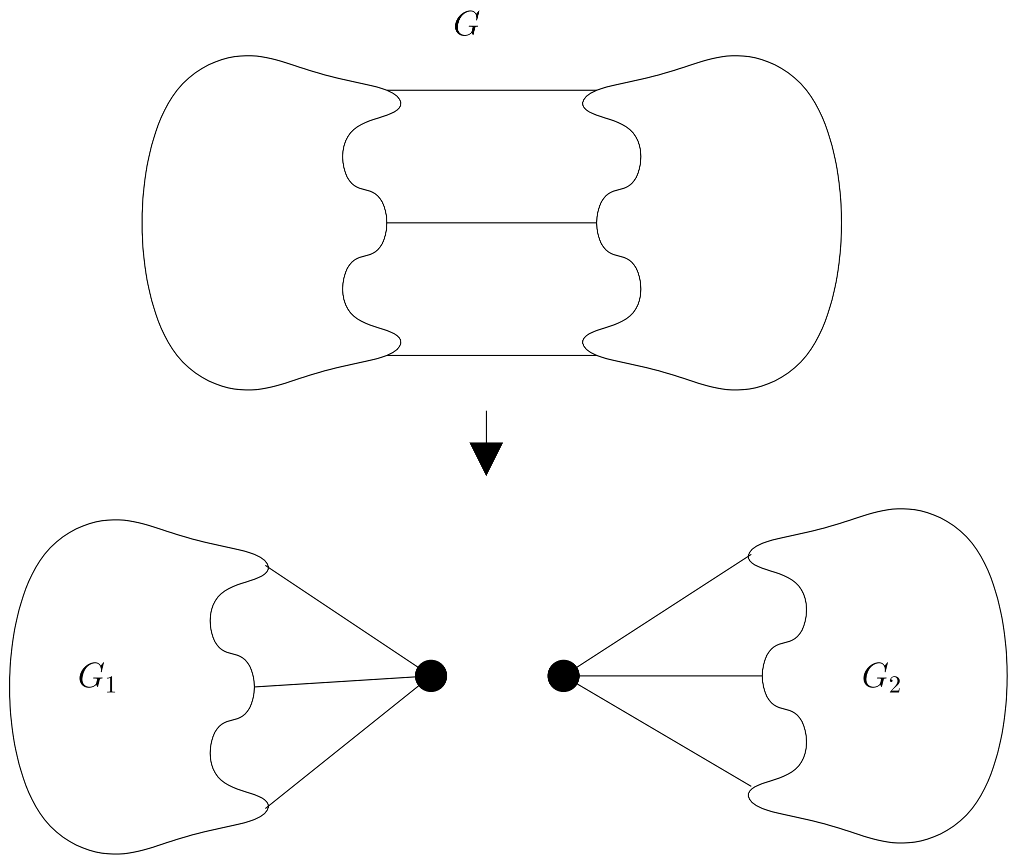
7 Using theorem to solve the topological containment problem for W_6

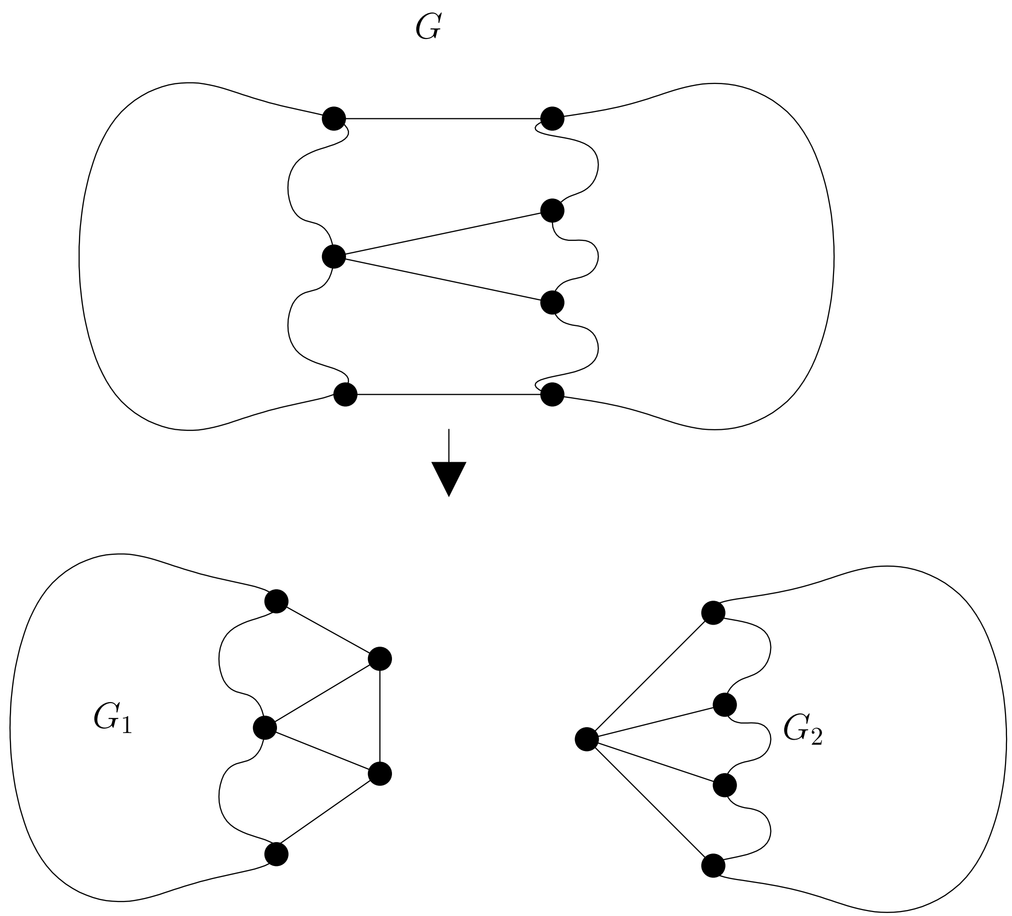
1. Find 3-connected components of G; apply algorithm recursively to each component.



2. Separate G along its 3-edge cutsets; apply algorithm recursively to each component.



3. Separate G along its 4-edge-cutsets; apply algorithm recursively to each component.

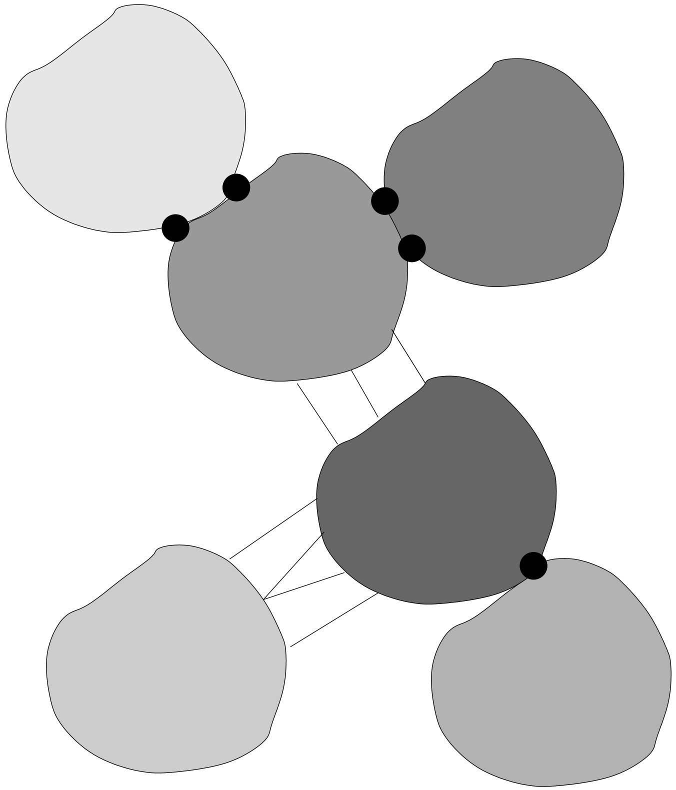


4. If G is topologically contained in Graph A, G has no W_6-subdivision; otherwise continue.

5. If some reduction R (either Reduction 1A or 2A) can be performed on G, let $G' = R(G)$. G contains a W_6-subdivision iff G' does.

6. If G has no vertex of degree at least 6, G has no W_6-subdivision. Otherwise, G contains a W_6-subdivision.

Structure of 'reduced' graphs with no W_6-subdivisions



8 Further results

- Result for W_6 strengthened and built on in similar way to obtain characterization for W_7
- As size of pattern graph H increases, so does the complexity of the characterization (and the proof!)

Theorem.

Let G be a 3-connected graph with at least 38 vertices. Suppose G has no internal 3 or 4-edge-cutsets, no internal $(1,1,1,1)$-cutsets, no type 1, 1a, 2, 2a, 3, 3a, 4, or 4a edge-vertex-cutsets, and is a graph on which Reductions 1A, 1B, 1C, 2A, 2B, 3, 4, 5, and 6 cannot be performed, for $k = 7$.

Then G has a W_7-subdivision if and only if G contains some vertex v_0 of degree at least 7.

- Wheel with 8 spokes?
- General characterization for graphs with no W_k-subdivisions
- Techniques used may be useful in solving the topological containment problem for pattern graphs other than wheels, for example, K_5