Fixed-Parameter Evolutionary Algorithms

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Computational Complexity of Evolutionary Algorithms

Theory of Evolutionary Algorithms

- Evolutionary algorithms are successful for many complex optimization problems.
- Rely on random decisions $\Rightarrow$ randomized algorithms
- Goal: Understand how and why they work
- Study the computational complexity of these algorithms on prominent examples

Runtime Analysis

Black Box Scenario:

- Measure the runtime T by the number of fitness evaluations.
- Studies consider time in dependence of the input to reach
  - An optimal solution.
  - A good approximation.

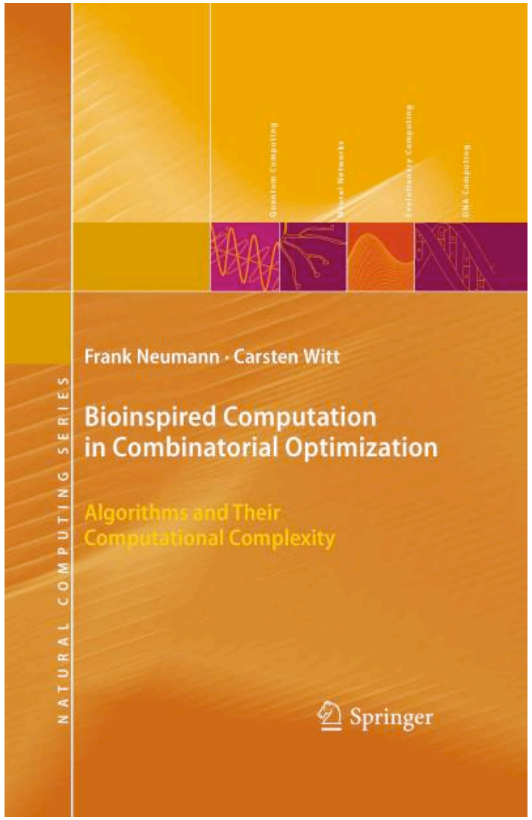
Interest:

- Expected number of fitness evaluations E[T].

Combinatorial Optimization

Analysis of runtime and approximation quality on combinatorial optimization problems, e. g.,

- sorting problems
- shortest path problems,
- subsequence problems,
- vertex cover,
- Eulerian cycles,
- minimum (multi)-cuts,
- minimum spanning trees,
- maximum matchings,
- partition problem,
- set cover problem,
- . . .



Book available at www.bioinspiredcomputation.com

Understand the behavior of bio-inspired computation on “natural” examples

Fixed Parameter Evolutionary Algorithms

- What makes a problem hard for an EA?

- Consider an additional parameter k to measure "hardness" of an instance
- Fixed parameter algorithm runs in time O(f(k) poly(n))

- Fixed parameter evolutionary algorithm runs in expected time O(f(k) poly(n))

- Consider maximum leaf spanning trees and minimum vertex covers as initial examples

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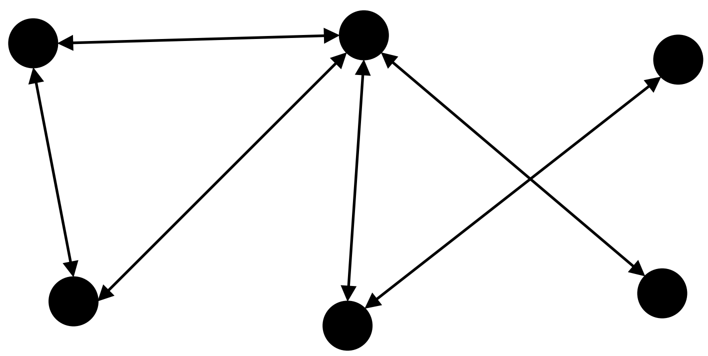
Maximum Leaf Spanning Trees

The Problem

The Maximum Leaf Spanning Tree Problem:

Given an undirected connected graph G=(V,E).

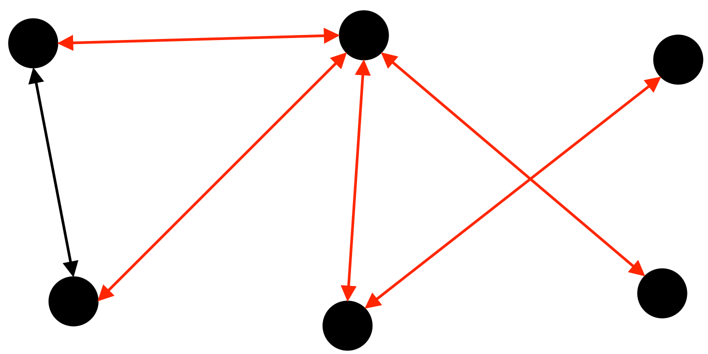


Find a spanning tree with a maximum number of leaves.

The Problem

The Maximum Leaf Spanning Tree Problem:

Given an undirected connected graph G=(V,E).



Find a spanning tree with a maximum number of leaves.

NP-hard, different classical FPT-studies

Two Evolutionary Algorithms

Algorithm 1 (Generic (1+1) EA)

1. *Choose a spanning tree of T uniformly at random.*
2. *Produce T' by swapping each edge of T independently with probability $1/m$.*
3. *If T' is a tree and $\ell(T') \geq \ell(T)$, set $T := T'$.*
4. *Go to 2.*

Algorithm 2 (Tree-Based (1+1) EA)

1. *Choose an arbitrary spanning tree T of G.*
2. *Choose S according to a Poisson distribution with parameter $\lambda = 1$ and perform sequentially S random edge-exchange operations to obtain a spanning tree T'. A random exchange operation applied to a spanning tree $\tilde{T}$ chooses an edge $e \in E \setminus \tilde{T}$ uniformly at random. The edge e is inserted and one randomly chosen edge of the cycle in $\tilde{T} \cup \{e\}$ is deleted.*
3. *If $\ell(T') \geq \ell(T)$, set $T := T'$.*
4. *Go to 2.*

Does the mutation operator make the difference between FPT and non-FPT runtime?

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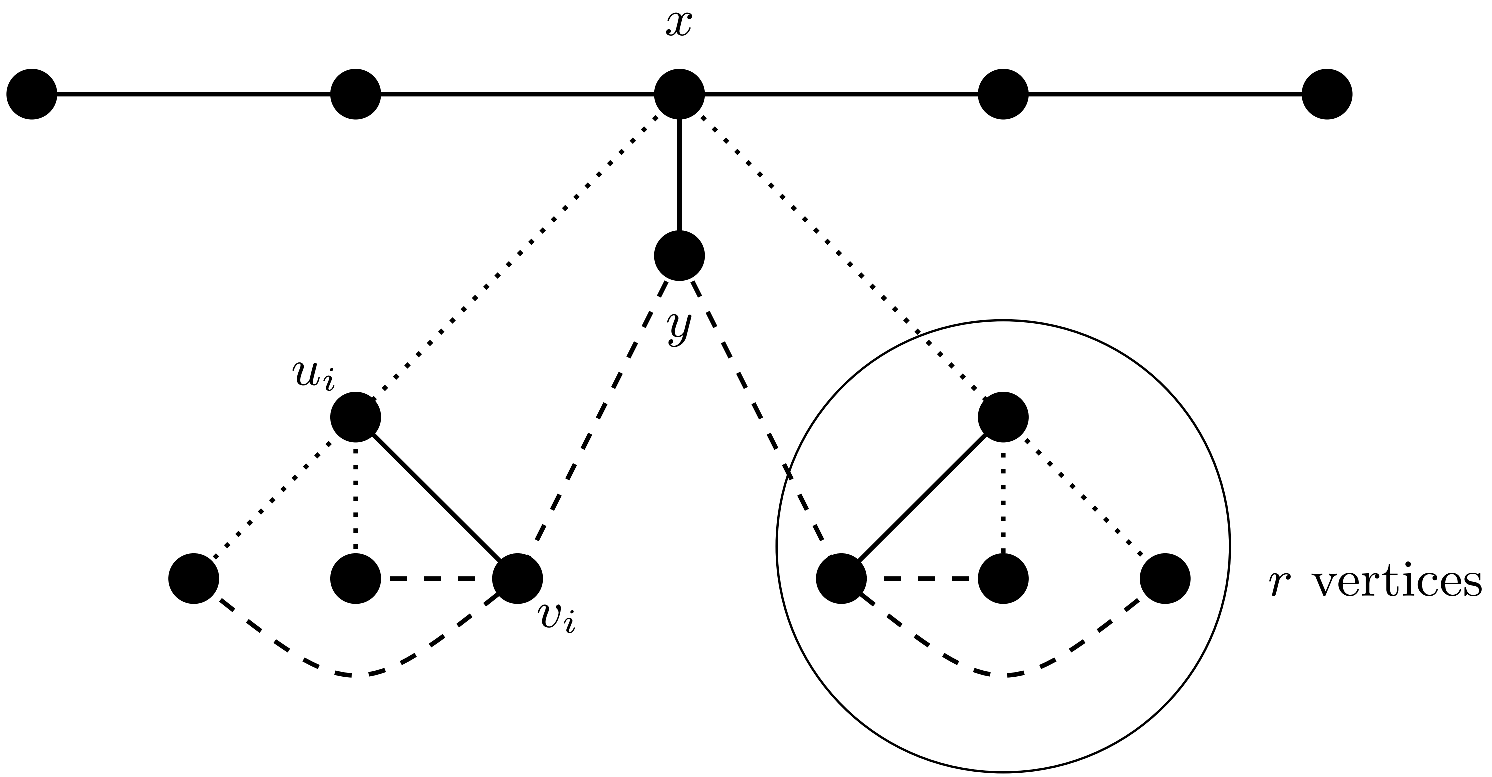
Local Optimum



Lower Bounds

Theorem 1. *The expected optimization time of Generic (1+1) EA on G_{loc} is lower bounded by $\left(\frac{m}{c}\right)^{2(r-2)}$ where c is an appropriate constant.*

Theorem 2. *The expected optimization time of Tree-Based (1+1) EA on G_{loc} is lower bounded by $(\frac{r-2}{c})^{r-2}$ where c is an appropriate constant.*

Idea for lower bounds:

Both algorithms may get stuck in local optimum.

For the Generic (1+1) EA it is less likely to escape local optimum as it often flips edges on the path.

Structural insights

Similar to Fellows, Lokshtanov, Misra, Mnich, Rosamond, Saurabh (2009)

Lemma 2. *Any connected graph G on n nodes and with a maximum number of k leaves in any spanning tree has at most $n+5k^2-7k$ edges and at most $10k-14$ nodes of degree at least three.*

Proof idea:

- Let T be a maximum leaf spanning tree with k leaves.
- Let P_0 be the set of all leaves and all nodes of degree at least three in T.
- Let P be the set of nodes that are of distance at most 2 (w. r. t. to T) to any node in P_0 and let Q be the set of remaining nodes.
- Show: all nodes of Q have degree 2 in G.
- Implies: Number of nodes in P is at most 10k-14
- No node has degree greater than k which implies bound on the number of edges.

Upper Bound

Theorem 3. *If the maximal number of leaf nodes in any spanning tree of G is k, then Algorithm 2 finds an optimal solution in expected time $O(2^{15k^2 \log k})$.*

Proof Idea:

- We call an edge distinguished if it is adjacent to at least one node of degree at least 3 in G.
- Number of distinguished edges on any cycle is at most 20k-28.
- Total number of edges in G: m <= n+5k^2–7k
- Probability to introduce a specific non-chosen distinguished edge is at least $1/(m-(n-1)) \geq 1/5k^2$
- Show: Length of created cycle is at most 20k.
- Probability to remove edge of the cycle that does not belong to optimal solution is at least $1/20k$

Proof Upper bound (continued)

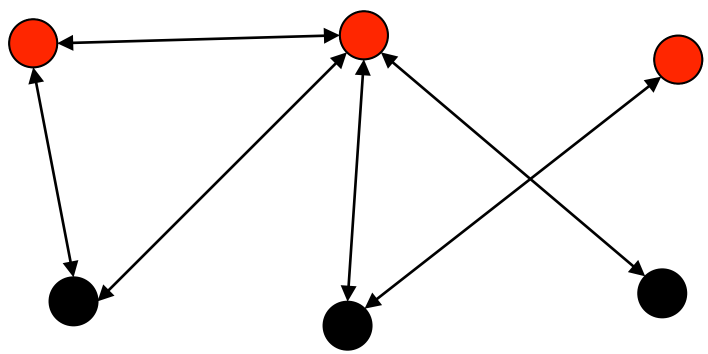
- Probability to obtain a specific spanning tree that can be obtained by an edge-swap is at least
  $$1/(20k \cdot 5k^2)$$
- Probability to produce optimal spanning tree which has distance $r \leq 5k^2$ is at least
  $$r! \cdot \frac{1}{er!} \cdot \left(\frac{1}{5k^2} \cdot \frac{1}{20k}\right)^r \geq \frac{1}{e}\left(\frac{1}{100k^3}\right)^{5k^2} \geq \frac{1}{e}\left(\frac{1}{100}\right)^{5k^2}\left(\frac{1}{k}\right)^{3 \cdot 5k^2},$$
- Implies that expected time to get maximum leaf spanning tree is at most $O(2^{15k^2 \log k})$ $\square$

The Minimum Vertex Cover Problem

The Problem

The Vertex Cover Problem:

Given an undirected graph G=(V,E).



Find a minimum subset of vertices such that each edge is covered at least once.

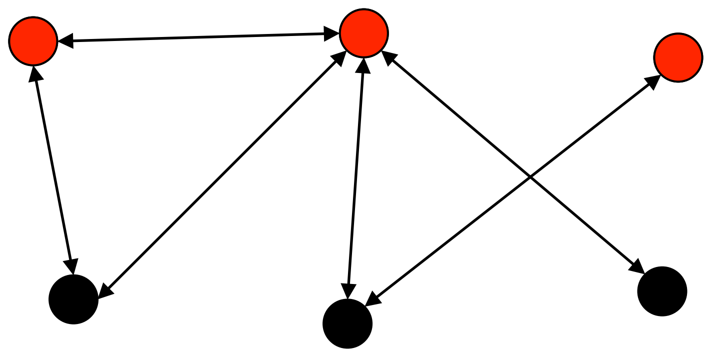
NP-hard, several 2-approximation algorithms.

Simple single-objective evolutionary algorithms fail!!!

The Problem

The Vertex Cover Problem:

Given an undirected graph G=(V,E).



Integer Linear Program (ILP)

$$\begin{aligned} &\min \textstyle\sum_{i=1}^{n} x_i \\ &\text{s.t. } x_i + x_j \geq 1 \qquad \forall \{i,j\} \in E \\ &x_i \in \{0,1\} \end{aligned}$$

Linear Program (LP)

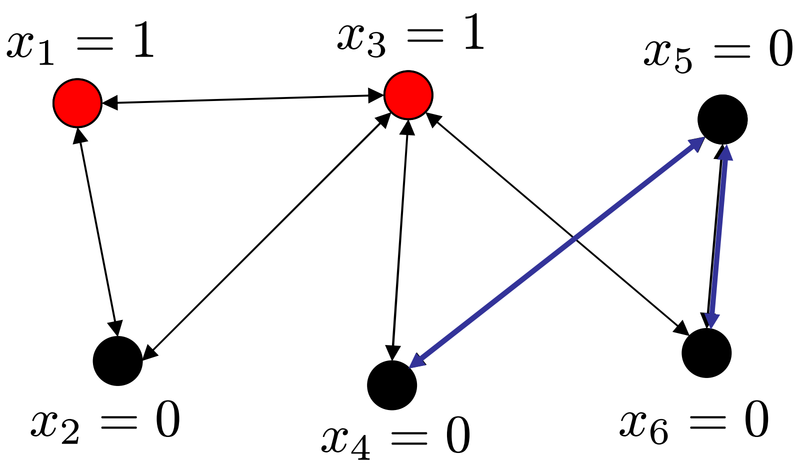
$$\begin{aligned} &\min \textstyle\sum_{i=1}^{n} x_i \\ &\text{s.t. } x_i + x_j \geq 1 \qquad \forall \{i,j\} \in E \\ &x_i \in [0,1] \end{aligned}$$

Decision problem:
Is there a set of vertices of size at most k covering all edges?

Our parameter: Value of an optimal solution (OPT)

Evolutionary Algorithm

Representation: Bitstrings of length n



Minimize fitness function:

$$f_1(x) = (|x|_1, |U(x)|)$$
$$f_1(x) = (2, 2)$$

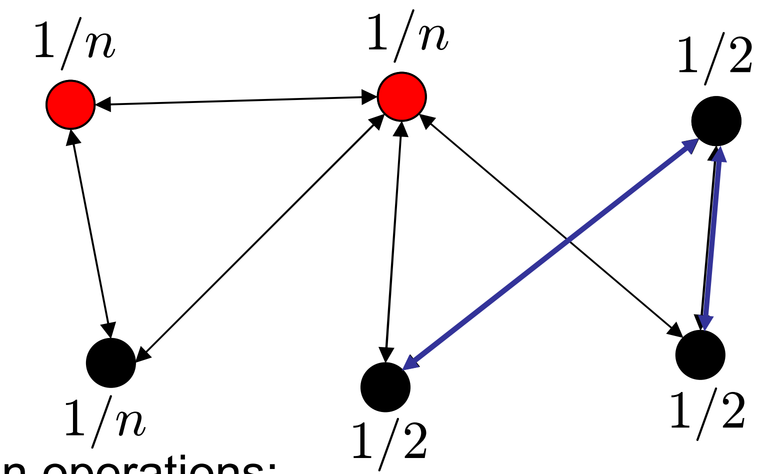
$$f_2(x) = (|x|_1, LP(x))$$
$$f_2(x) = (2, 1)$$

$U(x)$: Edges not covered by x

$G(x) = G(V, U(x))$

$LP(x)$: value of LP applied to $G(x)$

Evolutionary Algorithm



Two mutation operations:

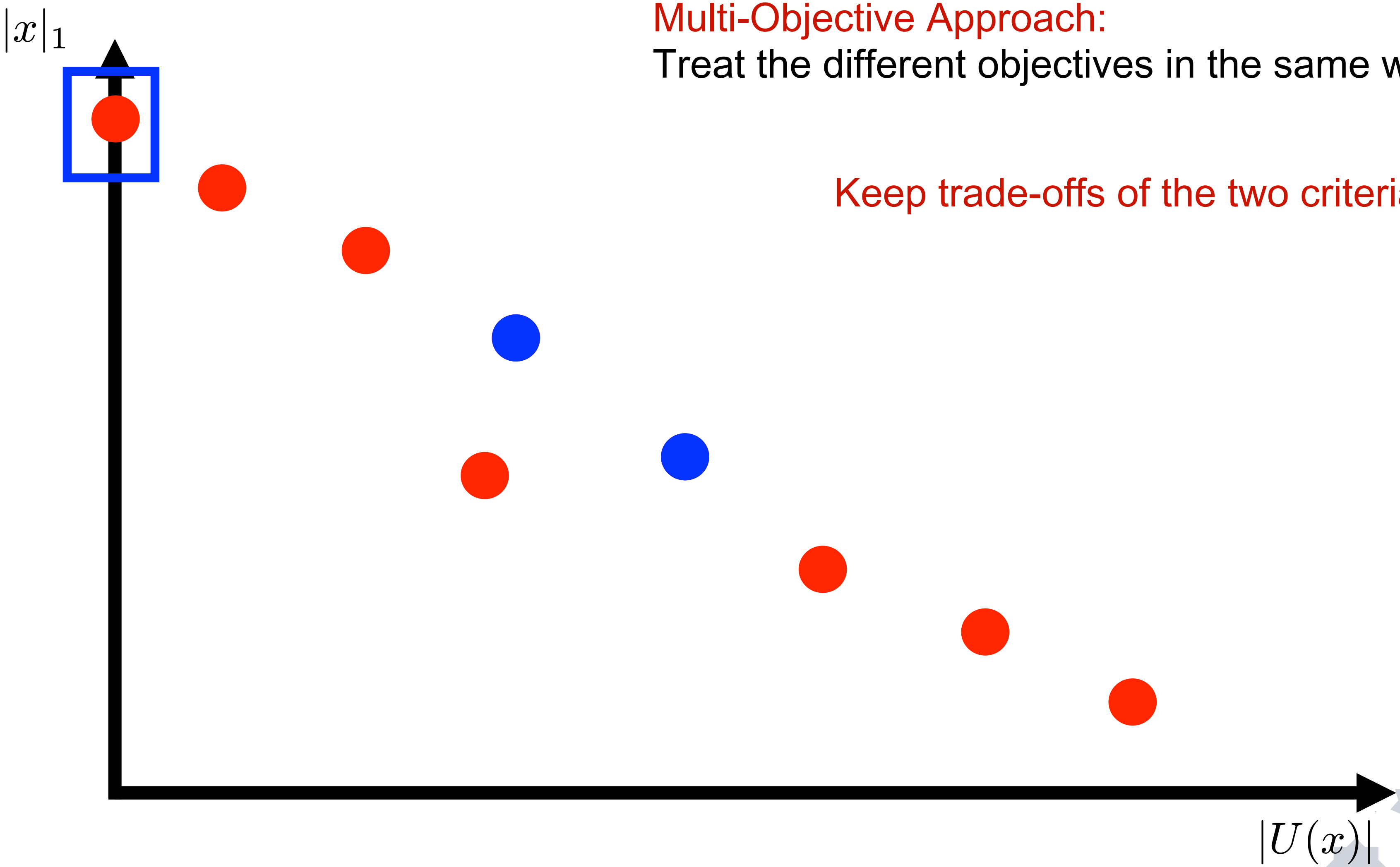
1. Standard bit mutation with probability $1/n$

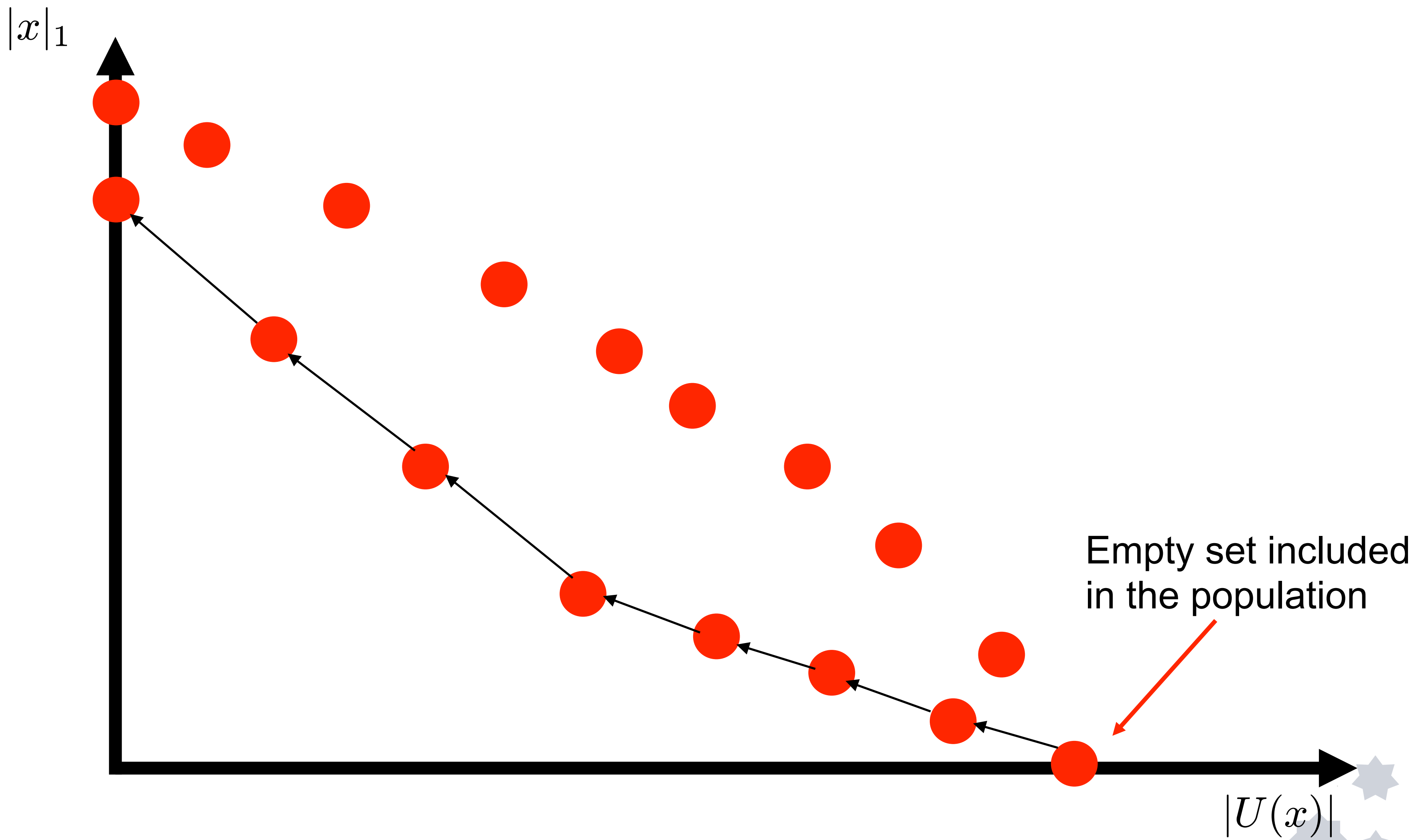
2. Mutation probability $1/2$ for vertices adjacent to edges of $U(x)$. Otherwise mutation probability $1/n$.

Decide uniformly at random which operator to use in next iteration

Multi-Objective Approach:
Treat the different objectives in the same way

Keep trade-offs of the two criteria




$|x|_1$
Empty set included
in the population
$|U(x)|$

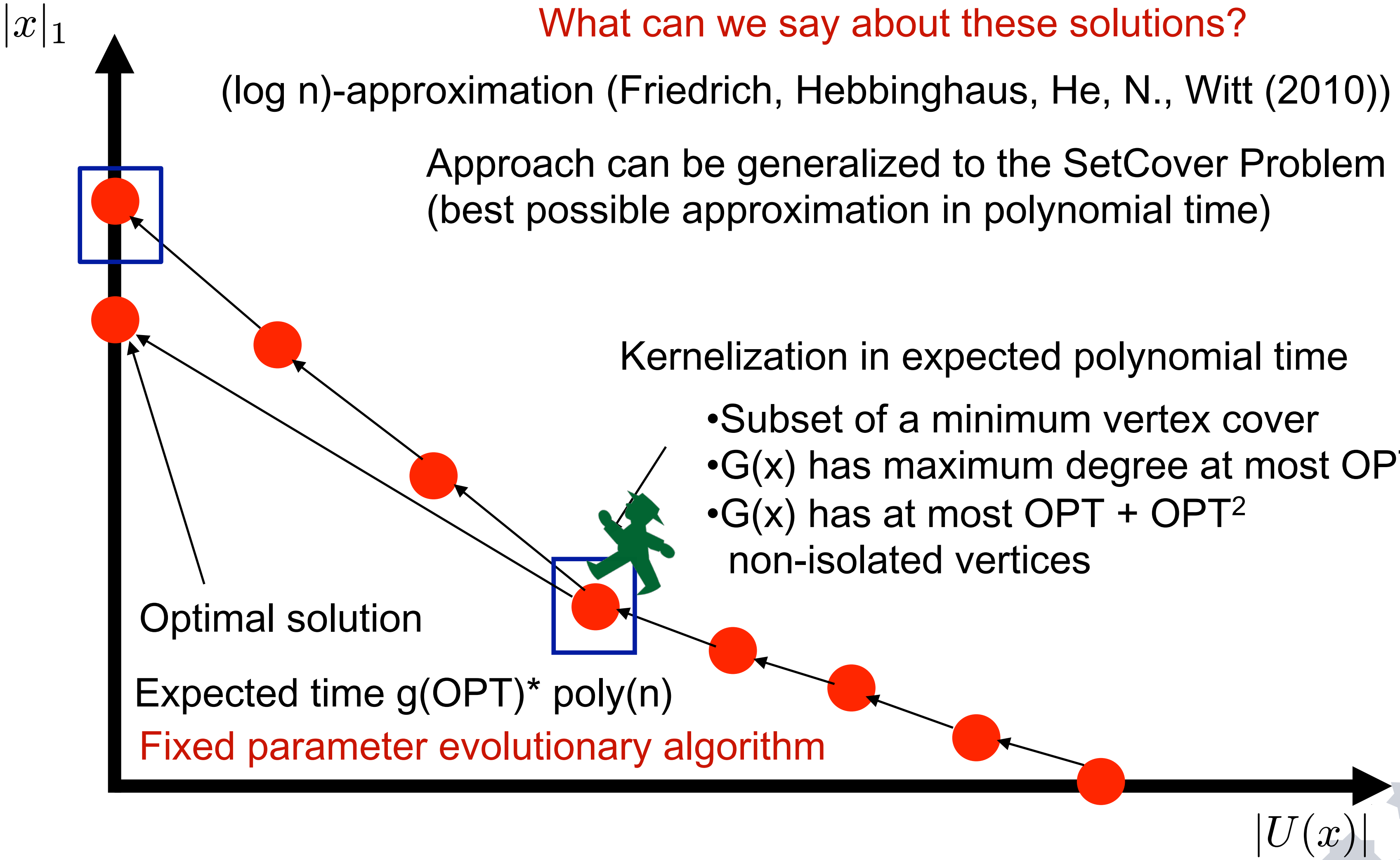
What can we say about these solutions?

(log n)-approximation (Friedrich, Hebbinghaus, He, N., Witt (2010))

Approach can be generalized to the SetCover Problem
(best possible approximation in polynomial time)

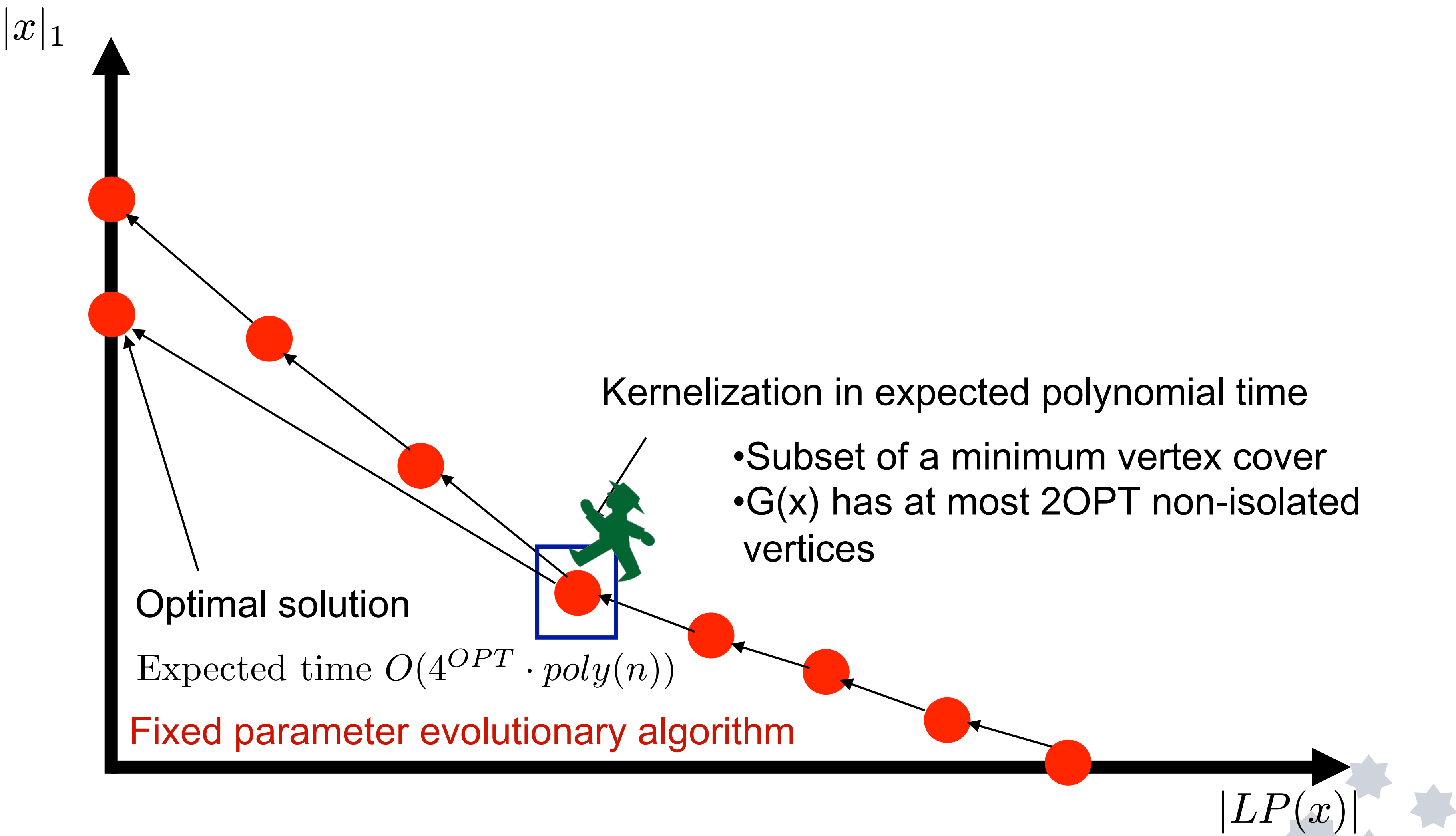
Kernelization in expected polynomial time

- Subset of a minimum vertex cover
- G(x) has maximum degree at most OPT
- G(x) has at most OPT + OPT^2 non-isolated vertices



Expected time g(OPT)* poly(n)

Fixed parameter evolutionary algorithm



$|x|_1$
Kernelization in expected polynomial time
•Subset of a minimum vertex cover
•G(x) has at most 2OPT non-isolated vertices
Optimal solution
Expected time $O(4^{OPT} \cdot poly(n))$
Fixed parameter evolutionary algorithm
$|LP(x)|$

Linear Programming

Combination with Linear Programming

- LP-relaxation is half integral, i.e.

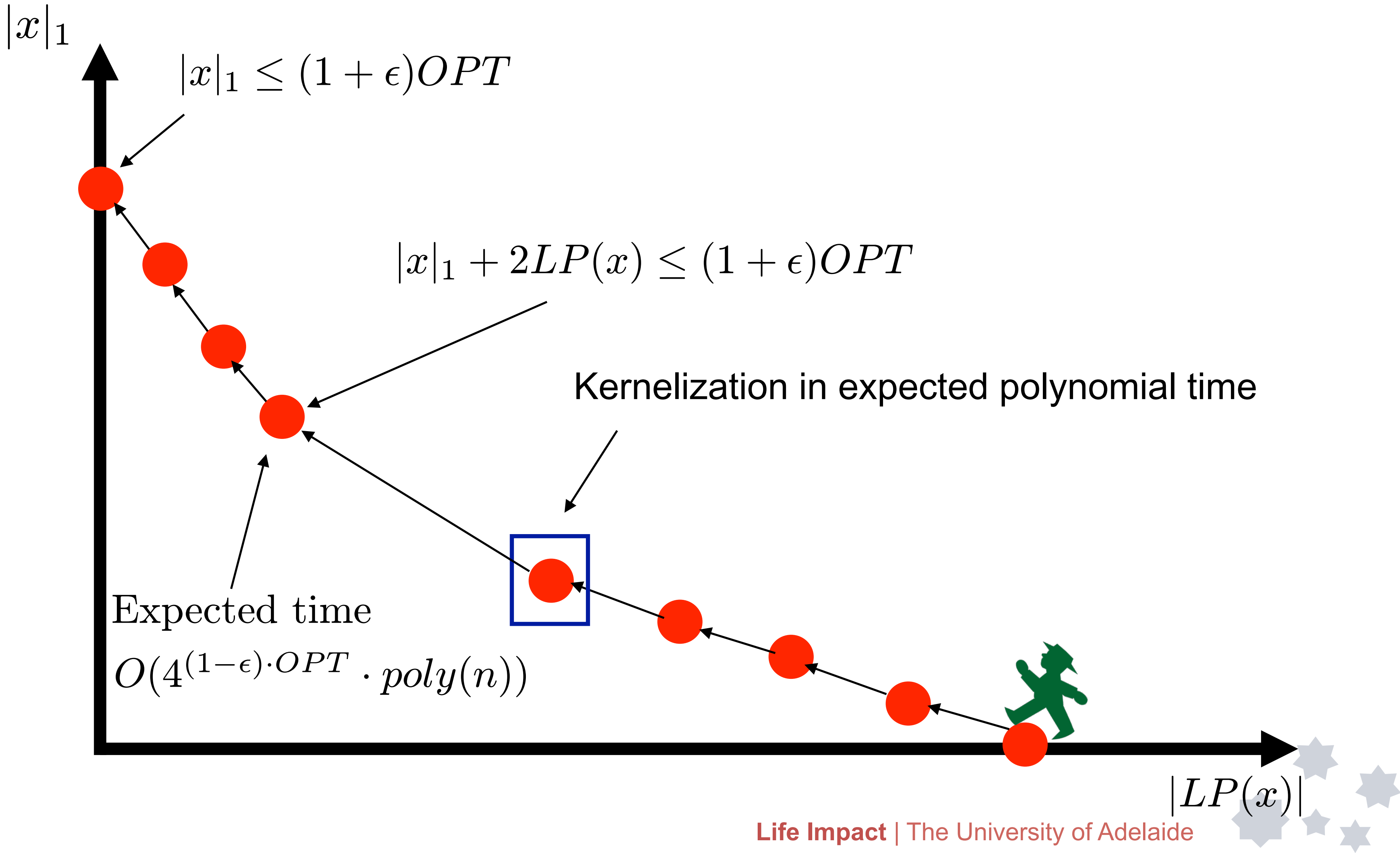
  $x_i \in \{0, 1/2, 1\}, 1 \leq i \leq n$

Theorem (Nemhauser, Trotter (1975)):
Let x^* be an optimal solution of the LP. Then there is a minimum vertex cover that contains all vertices v_i where $x_i^* = 1$.

Lemma:
All search points x with $LP(x) = LP(0^n) - |x|_1$ are Pareto optimal.
They can be extended to minimum vertex cover by selecting additional vertices.

Can we also say something about approximations?

Approximations



Summary

- Evolutionary algorithms are successful for many complex optimization problems.
- Goal is to get a better theoretical understanding.
- There are some nice results for combinatorial optimization.
- Using parameterized analysis looks very promising.

Thank you!