Worst-Case Optimal Redistribution of VCG Payments in Multi-Unit Auctions

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Abstract

For allocation problems with one or more items, the well-known Vickrey-Clarke-Groves (VCG) mechanism (aka. Clarke mechanism, Generalized Vickrey Auction) is efficient, strategy-proof, individually rational, and does not incur a deficit. However, it is not (strongly) budget balanced: generally, the agents' payments will sum to more than 0. We study mechanisms that *redistribute* some of the VCG payments back to the agents, while maintaining the desirable properties of the VCG mechanism. Our objective is to come as close to budget balance as possible in the *worst* case (so that we do not require a prior). For auctions with multiple indistinguishable units in which marginal values are nonincreasing, we derive a mechanism that is optimal in this sense. We also derive an optimal mechanism for the case where we drop the non-deficit requirement. Finally, we show that if marginal values are not required to be nonincreasing, then the original VCG mechanism is worst-case optimal.

1 Introduction

In resource allocation problems, we want to allocate the resources (or *items*) to the agents that value them the most. Unfortunately, agents' valuations are private knowledge, and self-interested agents will lie about their valuations if this is to their benefit. One solution is to *auction* off the items, possibly in a *combinatorial* auction where agents can bid on bundles of items. There exist ways of determining the payments that the agents make in such an auction that incentivizes the agents to report their true valuations—that is, the payments make the auction *strategy-proof*. One very general way of doing so is to use the VCG mechanism [30, 6, 16]. (In this paper, "the VCG mechanism" refers to the Clarke mechanism, not to any other Groves mechanism. In the specific context of auctions, the VCG mechanism is also known as the Generalized Vickrey Auction.[1])

[1]The phrase "VCG mechanisms" is sometimes used to refer to the class of all Groves mechanisms, which includes the Clarke mechanism. The new mechanisms that we propose in this paper are in fact also Groves

Besides strategy-proofness, the VCG mechanism has several other nice properties in the context of resource allocation problems. (Throughout, we assume *free disposal*, that is, not all items need to be allocated to the agents.) It is *efficient*: the chosen allocation always maximizes the sum of the agents' valuations. It is also *(ex-post) individually rational*: participating in the mechanism never makes an agent worse off than not participating. Finally, it has a *non-deficit* property: the sum of the agents' payments is always nonnegative.

In many settings, another property that would be desirable is *(strong) budget balance*, meaning that the payments sum to exactly 0. Suppose the agents are trying to distribute some resources among themselves that do not have a previous owner. For example, the agents may be trying to allocate the right to use a shared good on a given day. Or, the agents may be trying to allocate a resource that they have collectively constructed, discovered, or otherwise obtained. If the agents use an auction to allocate these resources, and the sum of the agents' payments in the auction is positive, then this surplus payment must leave the system of the agents (for example, the agents must give the money to an outside party, or burn it). Naïve redistribution of the surplus payment (*e.g.* each of the n agents receives $1/n$ of the surplus) will generally result in a mechanism that is not strategy-proof (*e.g.* in a Vickrey auction, the second-highest bidder would want to increase her bid to obtain a larger redistribution payment). Unfortunately, the VCG mechanism is not budget balanced: typically, there is surplus payment. Unfortunately, in general settings, it is in fact impossible to design mechanisms that satisfy budget balance in addition to the other desirable properties [21, 15, 14, 26].

In light of this impossibility result, several authors have obtained budget balance by sacrificing some of the other desirable properties [3, 9, 27, 8]. Another approach that is perhaps preferable is to use a mechanism that is "more" budget balanced than the VCG mechanism, and maintains all the other desirable properties. One way of trying to design such a mechanism is to redistribute some of the VCG payment back to the agents in a way that will not affect the agents' incentives (so that strategy-proofness is maintained), and that will maintain the other properties. In 2006, Cavallo [4] pursued exactly this idea, and designed a mechanism that redistributes a large amount of the total VCG payment while maintaining all of the other desirable properties of the VCG mechanism. For example, in a single-item auction (where the VCG mechanism coincides with the second-price sealed-bid auction), the amount redistributed to bidder i by Cavallo's mechanism is $1/n$ times the second-highest bid among bids *other than* i's bid. The total redistributed is at most the second-highest bid overall, and the redistribution to agent i does not affect i's incentives because it does not depend on i's own bid. For general settings, Cavallo's mechanism considers how small an agent could make the total VCG payment by changing her bid (the resulting minimal total VCG payment is never greater than the actual total VCG payment), and redistributes $1/n$ of that to the agent (and therefore satisfies the non-deficit property).[2]

mechanisms.

[2]In this mechanism, as well as in the mechanisms introduced in this paper, an agent may end up making a negative payment (receiving a positive amount) overall. For example, an agent may not win anything and still receive a positive redistribution payment. Under the restriction that payments must be nonnegative, several authors have proposed mechanisms that maximize the agents' combined utility after deducting the payments, in expectation [20, 5].

In this paper, we extend Cavallo's technique in a limited setting. We study allocation settings where there are multiple indistinguishable units of a single good, and each agent's valuation function is concave—that is, agents have nonincreasing marginal values. For this setting, Cavallo's mechanism coincides with a mechanism proposed by Bailey in 1997 [3]. (For the case of a single item, the same mechanism has also been proposed by Porter *et al.* [28].) Cavallo's mechanism and Bailey's mechanism are in fact the same in any setting under which VCG satisfies *revenue monotonicity*, for the following reason. Bailey's mechanism redistributes to each agent $1/n$ of the total VCG payment that would result if this agent were removed from the auction. If the total VCG payment is nondecreasing in agents, then, when computing payments under Cavallo's mechanism, the bid that would minimize the total VCG payment is the one that has a valuation of 0 for everything, which is equivalent to not participating in the auction. Hence, Cavallo's mechanism results in the same redistribution payment as Bailey's. It is well-known that in general, the VCG mechanism does *not* satisfy this revenue monotonicity criterion [2, 7, 31, 32, 33] (this is in fact true for a much wider class of mechanisms [29]). However, in more restricted settings, such as the ones considered in this paper, revenue monotonicity often holds.

From Section 2 to Section 9, we consider a slightly simpler setting where all agents have *unit demand*, *i.e.* they want only a single unit. We propose the family of *linear* VCG redistribution mechanisms. All mechanisms in this family are efficient, strategy-proof, individually rational, and never incur a deficit. The family includes the Bailey-Cavallo mechanism as a special case (with the caveat that Bailey's and Cavallo's mechanisms can be applied in more general settings). We then provide an optimization model for finding the optimal mechanism inside the family, based on worst-case analysis. We convert this optimization model into a linear program. Both numerical and analytical solutions of this linear program are provided, and the resulting mechanism shows significant improvement over the Bailey-Cavallo mechanism (in the worst case). For example, for the problem of allocating a single unit, when the number of agents is 10, the resulting mechanism always redistributes more than 98% of the total VCG payment back to the agents (whereas the Bailey-Cavallo mechanism redistributes only 80% in the worst case). Finally, we prove that this mechanism is in fact optimal among *all* anonymous deterministic mechanisms (even nonlinear ones) that satisfy the desirable properties.

Around the same time, the same mechanism (in the unit demand setting only) has been independently derived by Moulin [24].[3] Moulin actually pursues a different objective (also based on worst-case analysis): whereas our objective is to maximize the fraction of VCG payments that are redistributed, Moulin tries to minimize the overall payments from agents as a fraction of efficiency. It turns out that the resulting mechanisms are the same. However, for our objective, the optimal mechanism does not change even if the individual rationality requirement is dropped, while for Moulin's objective, dropping individual rationality does change the optimal mechanism (but only if there are multiple units).

In Section 9, we drop the non-deficit requirement and solve for the mechanism that is as close to budget balance as possible (in the worst case). This mechanism is in fact

[3]We thank Rakesh Vohra for pointing us to Moulin's working paper.

closer to budget balance than the best non-deficit mechanism.[4]

In Section 10, we consider the more general setting where the agents do not necessarily have unit demand, but have nonincreasing marginal values. We generalize the optimal redistribution mechanism to this setting (both with and without the individual rationality constraint, and both with or without the non-deficit constraint). In each case, the worst-case performance is the same as for the unit demand setting.

Finally, in Section 11, we consider multi-unit auctions without restrictions on the agents' valuations—marginal values may increase. Here, we show a negative result: when there are at least two units, no redistribution mechanism performs better (in the worst case) than the original VCG mechanism (redistributing nothing).

2 Problem Description

From this section to Section 9, we consider only the unit demand setting. (Units are indistinguishable throughout the paper.) Let n denote the number of agents, and let m denote the number of units. We only consider the case where $m < n$ (otherwise the problem becomes trivial in the unit demand setting). We also assume that m and n are always known. This assumption is not harmful: in environments where anyone can join the auction, running a redistribution mechanism is typically not a good idea anyway, because everyone would want to join to collect part of the redistribution.

In the unit demand setting, an agent's marginal value for any unit after the first is zero. Hence, the agent's valuation function corresponds to a single value, which is her valuation for having at least one unit.

Let the set of agents be $\{a_1, a_2, \ldots, a_n\}$, where a_i is the agent with ith highest report value $\hat{v}_i$—that is, we have $\hat{v}_1 \geq \hat{v}_2 \geq \ldots \geq \hat{v}_n \geq 0$. Let v_i denote the true value of a_i. Given that the mechanism is strategy-proof, we can assume $v_i = \hat{v}_i$.

Under the VCG mechanism, each agent among $a_1, \ldots, a_m$ wins a unit, and pays $\hat{v}_{m+1}$ for this unit. Thus, the total VCG payment equals $m\hat{v}_{m+1}$. When $m = 1$, this is the second-price or Vickrey auction.

We modify the mechanism as follows. After running the original VCG mechanism, the center returns to each agent a_i some amount z_i, agent a_i's *redistribution payment*. We do not allow z_i to depend on $\hat{v}_i$; because of this, a_i's incentives are unaffected by this redistribution payment, and the mechanism remains strategy-proof.

3 Linear VCG Redistribution Mechanisms

We are now ready to introduce the family of linear VCG redistribution mechanisms. Such a mechanism is defined by a vector of constants $c_0, c_1, \ldots, c_{n-1}$. The amount that the mechanism returns to agent a_i is $z_i = c_0 + c_1\hat{v}_1 + c_2\hat{v}_2 + \ldots + c_{i-1}\hat{v}_{i-1} + c_i\hat{v}_{i+1} + \ldots + c_{n-1}\hat{v}_n$. That is, an agent receives c_0, plus c_1 times the highest bid *other* than the agent's own bid, plus c_2 times the second-highest other bid, *etc.* The mechanism

[4] Moulin [24] also notes that dropping the non-deficit requirement can bring us closer to budget balance, but does not solve for the optimal mechanism.

is strategy-proof, because for all i, z_i is independent of $\hat{v}_i$. Also, the mechanism is anonymous and efficient.

It is helpful to see the entire list of redistribution payments:

$$
\begin{aligned}
&z_1 = c_0 + c_1\hat{v}_2 + c_2\hat{v}_3 + c_3\hat{v}_4 + \ldots + c_{n-2}\hat{v}_{n-1} + c_{n-1}\hat{v}_n \\
&z_2 = c_0 + c_1\hat{v}_1 + c_2\hat{v}_3 + c_3\hat{v}_4 + \ldots + c_{n-2}\hat{v}_{n-1} + c_{n-1}\hat{v}_n \\
&z_3 = c_0 + c_1\hat{v}_1 + c_2\hat{v}_2 + c_3\hat{v}_4 + \ldots + c_{n-2}\hat{v}_{n-1} + c_{n-1}\hat{v}_n \\
&z_4 = c_0 + c_1\hat{v}_1 + c_2\hat{v}_2 + c_3\hat{v}_3 + \ldots + c_{n-2}\hat{v}_{n-1} + c_{n-1}\hat{v}_n \\
&\vdots \\
&z_i = c_0 + c_1\hat{v}_1 + c_2\hat{v}_2 + \ldots + c_{i-1}\hat{v}_{i-1} + c_i\hat{v}_{i+1} + \ldots + c_{n-1}\hat{v}_n \\
&\vdots \\
&z_{n-2} = c_0 + c_1\hat{v}_1 + c_2\hat{v}_2 + c_3\hat{v}_3 + \ldots + c_{n-2}\hat{v}_{n-1} + c_{n-1}\hat{v}_n \\
&z_{n-1} = c_0 + c_1\hat{v}_1 + c_2\hat{v}_2 + c_3\hat{v}_3 + \ldots + c_{n-2}\hat{v}_{n-2} + c_{n-1}\hat{v}_n \\
&z_n = c_0 + c_1\hat{v}_1 + c_2\hat{v}_2 + c_3\hat{v}_3 + \ldots + c_{n-2}\hat{v}_{n-2} + c_{n-1}\hat{v}_{n-1}
\end{aligned}
$$

Not all choices of the constants $c_0, \ldots, c_{n-1}$ produce a mechanism that is individually rational, and not all choices of the constants produce a mechanism that never incurs a deficit. Hence, to obtain these properties, we need to place some constraints on the constants.

To satisfy the individual rationality criterion, each agent's utility should always be nonnegative. An agent that does not win a unit obtains a utility that is equal to the agent's redistribution payment. An agent that wins a unit obtains a utility that is equal to the agent's valuation for the unit, minus the VCG payment $\hat{v}_{m+1}$, plus the agent's redistribution payment.

Consider agent a_n, the agent with the lowest bid. Since this agent does not win an item ($m < n$), her utility is just her redistribution payment z_n. Hence, for the mechanism to be individually rational, the c_i must be such that z_n is always nonnegative. If the c_i have this property, then it actually follows that z_i is nonnegative for *every* i, for the following reason. Suppose there exists some $i < n$ and some vector of bids $\hat{v}_1 \geq \hat{v}_2 \geq \ldots \geq \hat{v}_n \geq 0$ such that $z_i < 0$. Then, consider the bid vector that results from replacing $\hat{v}_j$ by $\hat{v}_{j+1}$ for all $j \geq i$, and letting $\hat{v}_n = 0$. If we omit $\hat{v}_n$ from this vector, the same vector results that results from omitting $\hat{v}_i$ from the original vector. Therefore, a_n's redistribution payment under the new vector should be the same as a_i's redistribution payment under the old vector—but this payment is negative.

If all redistribution payments are always nonnegative, then the mechanism must be individually rational (because the VCG mechanism is individually rational, and the redistribution payment only increases an agent's utility). Therefore, the mechanism is individually rational if and only if for any bid vector, $z_n \geq 0$.

To satisfy the non-deficit criterion, the sum of the redistribution payments should be less than or equal to the total VCG payment. So for any bid vector $\hat{v}_1 \geq \hat{v}_2 \geq \ldots \geq \hat{v}_n \geq 0$, the constants c_i should make $z_1 + z_2 + \ldots + z_n \leq m\hat{v}_{m+1}$.

We define the family of linear VCG redistribution mechanisms to be the set of all redistribution mechanisms corresponding to constants c_i that satisfy the above constraints (so that the mechanisms will be individually rational and have the non-deficit property). It turns out that some of the c_i always need to be set to 0, as the following claim demonstrates.

Claim 1 *If $c_0, c_1, \ldots, c_{n-1}$ satisfy both the individual rationality and the non-deficit constraints, then $c_i = 0$ for $i = 0, \ldots, m$.*

Proof: First, let us prove that $c_0 = 0$. Consider the bid vector in which $\hat{v}_i = 0$ for all i. To obtain individual rationality, we must have $c_0 \geq 0$. To satisfy the non-deficit constraint, we must have $c_0 \leq 0$. Thus we know $c_0 = 0$. Now, if $c_i = 0$ for all i, there is nothing to prove. Otherwise, let $j = min\{i | c_i \neq 0\}$. Assume that $j \leq m$. We recall that we can write the individual rationality constraint as follows: $z_n = c_0 + c_1\hat{v}_1 + c_2\hat{v}_2 + c_3\hat{v}_3 + \ldots + c_{n-2}\hat{v}_{n-2} + c_{n-1}\hat{v}_{n-1} \geq 0$ for any bid vector. Let us consider the bid vector in which $\hat{v}_i = 1$ for $i \leq j$ and $\hat{v}_i = 0$ for the rest. In this case $z_n = c_j$, so we must have $c_j \geq 0$. The non-deficit constraint can be written as follows: $z_1 + z_2 + \ldots + z_n \leq m\hat{v}_{m+1}$ for any bid vector. Consider the same bid vector as above. We have $z_i = 0$ for $i \leq j$, because for these bids, the jth highest other bid has value 0, so all the c_i that are nonzero are multiplied by 0. For $i > j$, we have $z_i = c_j$, because the jth highest other bid has value 1, and all lower bids have value 0. So the non-deficit constraint tells us that $c_j(n-j) \leq m\hat{v}_{m+1}$. Because $j \leq m$, $\hat{v}_{m+1} = 0$, so the right hand side is 0. We also have $n - j > 0$ because $j \leq m < n$. So $c_j \leq 0$. Because we have already established that $c_j \geq 0$, it follows that $c_j = 0$; but this is contrary to assumption. So $j > m$. ■

Incidentally, this claim also shows that if $m = n-1$, then $c_i = 0$ for all i. Thus, we are stuck with the VCG mechanism (more details in Claim 7). From here on, we only consider the case where $m < n - 1$.

We now give two examples of mechanisms in this family.

Example 1 (Bailey-Cavallo mechanism): Consider the mechanism corresponding to $c_{m+1} = \frac{m}{n}$ and $c_i = 0$ for all other i. Under this mechanism, each agent receives a redistribution payment of $\frac{m}{n}$ times the $(m+1)$th highest bid from another agent. Hence, $a_1, \ldots, a_{m+1}$ receive a redistribution payment of $\frac{m}{n}\hat{v}_{m+2}$, and the others receive $\frac{m}{n}\hat{v}_{m+1}$. Thus, the total redistribution payment is $(m+1)\frac{m}{n}\hat{v}_{m+2} + (n-m-1)\frac{m}{n}\hat{v}_{m+1}$. This redistribution mechanism is individually rational, because all the redistribution payments are nonnegative, and never incurs a deficit, because $(m+1)\frac{m}{n}\hat{v}_{m+2} + (n-m-1)\frac{m}{n}\hat{v}_{m+1} \leq n\frac{m}{n}\hat{v}_{m+1} = m\hat{v}_{m+1}$. (We note that for this mechanism to make sense, we need $n \geq m+2$.)

Example 2: Consider the mechanism corresponding to $c_{m+1} = \frac{m}{n-m-1}$, $c_{m+2} = -\frac{m(m+1)}{(n-m-1)(n-m-2)}$, and $c_i = 0$ for all other i. In this mechanism, each agent receives a redistribution payment of $\frac{m}{n-m-1}$ times the $(m+1)$th highest reported value from other agents, minus $\frac{m(m+1)}{(n-m-1)(n-m-2)}$ times the $(m+2)$th highest reported value from other agents. Thus, the total redistribution payment is $m\hat{v}_{m+1} - \frac{m(m+1)(m+2)}{(n-m-1)(n-m-2)}\hat{v}_{m+3}$. If $n \geq 2m+3$ (which is equivalent to $\frac{m}{n-m-1} \geq \frac{m(m+1)}{(n-m-1)(n-m-2)}$), then each agent always receives a nonnegative redistribution payment, thus the mechanism is individually rational. Also, the mechanism never incurs a deficit, because the total VCG payment is $m\hat{v}_{m+1}$, which is greater than the amount $m\hat{v}_{m+1} - \frac{m(m+1)(m+2)}{(n-m-1)(n-m-2)}\hat{v}_{m+3}$ that is redistributed.

Which of these two mechanisms is better? Is there another mechanism that is even better? This is what we study in the next section.

4 Optimal Redistribution Mechanisms

Among all linear VCG redistribution mechanisms, we would like to be able to identify the one that redistributes the greatest fraction of the total VCG payment.[5] This is not a well-defined notion: it may be that one mechanism redistributes more on some bid vectors, and another more on other bid vectors. We emphasize that we do not assume that a prior distribution over bidders' valuations is available, so we cannot compare them based on expected redistribution. Below, we study three well-defined ways of comparing redistribution mechanisms: best-case performance, dominance, and worst-case performance.

Best-case performance. One way of evaluating a mechanism is by considering the highest redistribution fraction that it achieves. Consider the previous two examples. For the first example, the total redistribution payment is $(m+1)\frac{m}{n}\hat{v}_{m+2} + (n-m-1)\frac{m}{n}\hat{v}_{m+1}$. When $\hat{v}_{m+2} = \hat{v}_{m+1}$, this is equal to the total VCG payment $m\hat{v}_{m+1}$. Thus, this mechanism redistributes 100% of the total VCG payment in the best case. For the second example, the total redistribution payment is $m\hat{v}_{m+1} - \frac{m(m+1)(m+2)}{(n-m-1)(n-m-2)}\hat{v}_{m+3}$. When $\hat{v}_{m+3} = 0$, this is equal to the total VCG payment $m\hat{v}_{m+1}$. Thus, this mechanism also redistributes 100% of the total VCG payment in the best case.

Moreover, there are actually infinitely many mechanisms that redistribute 100% of the total VCG payment in the best case—for example, any convex combination of the above two will redistribute 100% if both $\hat{v}_{m+2} = \hat{v}_{m+1}$ and $\hat{v}_{m+3} = 0$.

Dominance. Inside the family of linear VCG redistribution mechanisms, we say one mechanism *dominates* another mechanism if the first one redistributes at least as much as the other for *any* bid vector. For the previous two examples, neither dominates the other, because they each redistribute 100% in different cases. It turns out that there is no mechanism in the family that dominates all other mechanisms in the family. For suppose such a mechanism exists. Then, it should dominate both examples above. Consider the *remaining* VCG payment (the VCG payment failed to be redistributed). The remaining VCG payment of the dominant mechanism should be 0 whenever $\hat{v}_{m+2} = \hat{v}_{m+1}$ or $\hat{v}_{m+3} = 0$. Now, the remaining VCG payment is a linear function of the $\hat{v}_i$ (linear redistribution), and therefore also a polynomial function. The above implies that this function can be written as $(\hat{v}_{m+2} - \hat{v}_{m+1})(\hat{v}_{m+3})P(\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n)$, where P is a polynomial function. But since the function must be linear (has degree at most 1), it follows that $P = 0$. Thus, a dominant mechanism would always redistribute all of the VCG payment, which is not possible. (If it were possible, then our worst-case optimal redistribution mechanism would also always redistribute all of the VCG payment, and we will see later that it does not.)

Worst-case performance. Finally, we can evaluate a mechanism by considering the lowest redistribution fraction that it guarantees. For the first example, the total redistribution payment is $(m+1)\frac{m}{n}\hat{v}_{m+2} + (n-m-1)\frac{m}{n}\hat{v}_{m+1}$, which is greater than or equal to $(n-m-1)\frac{m}{n}\hat{v}_{m+1}$. In the worst case, which is when $\hat{v}_{m+2} = 0$,

[5] The fraction redistributed seems a natural criterion to use. One good property of this criterion is that it is scale-invariant: if we multiply all bids by the same positive constant (for example, if we change the units by re-expressing the bids in euros instead of dollars), we would not want the behavior of our mechanism to change.

the fraction redistributed is $\frac{n-m-1}{n}$. For the second example, the total redistribution payment is $m\hat{v}_{m+1} - \frac{m(m+1)(m+2)}{(n-m-1)(n-m-2)}\hat{v}_{m+3}$, which is greater than or equal to $m\hat{v}_{m+1}(1 - \frac{(m+1)(m+2)}{(n-m-1)(n-m-2)})$. In the worst case, which is when $\hat{v}_{m+3} = \hat{v}_{m+1}$, the fraction redistributed is $1 - \frac{(m+1)(m+2)}{(n-m-1)(n-m-2)}$. Since we assume that the number of agents n and the number of units m are known, we can determine which example mechanism has better worst-case performance by comparing the two quantities. When $n = 6$ and $m = 1$, for the first example (Bailey-Cavallo mechanism), the fraction redistributed in the worst case is $\frac{2}{3}$, and for the second example, this fraction is $\frac{1}{2}$, which implies that for this pair of n and m, the first mechanism has better worst-case performance. On the other hand, when $n = 12$ and $m = 1$, for the first example, the fraction redistributed in the worst case is $\frac{5}{6}$, and for the second example, this fraction is $\frac{14}{15}$, which implies that this time the second mechanism has better worst-case performance.

In this paper, we compare mechanisms by the fraction of the total VCG payment that they redistribute in the worst case. This fraction is undefined when the total VCG payment is 0. To deal with this, technically, we define the worst-case redistribution fraction as the largest k so that the total amount redistributed is at least k times the total VCG payment, for all bid vectors. (Hence, as long as the total amount redistributed is at least 0 when the total VCG payment is 0, these cases do not affect the worst-case fraction.) This corresponds to the following optimization problem:

Maximize k (the fraction redistributed in the worst case)
Subject to:
For every bid vector $\hat{v}_1 \geq \hat{v}_2 \geq \ldots \geq \hat{v}_n \geq 0$
$z_n \geq 0$ (individual rationality)
$z_1 + z_2 + \ldots + z_n \leq m\hat{v}_{m+1}$ (non-deficit)
$z_1 + z_2 + \ldots + z_n \geq km\hat{v}_{m+1}$ (worst-case constraint)
We recall that $z_i = c_0 + c_1\hat{v}_1 + c_2\hat{v}_2 + \ldots + c_{i-1}\hat{v}_{i-1} + c_i\hat{v}_{i+1} + \ldots + c_{n-1}\hat{v}_n$

5 Transformation to Linear Programming

The optimization problem given in the previous section can be rewritten as a linear program, based on the following observations.

Claim 2 *The individual rationality constraint can be written as follows:* $\sum_{i=m+1}^{j} c_i \geq 0$ *for* $j = m+1, \ldots, n-1$.

Before proving this claim, we introduce the following lemma.

Lemma 1 *Given a positive integer* k *and a set of real constants* $s_1, s_2, \ldots, s_k$, *(*$s_1t_1 + s_2t_2 + \ldots + s_kt_k \geq 0$ *for any* $t_1 \geq t_2 \geq \ldots \geq t_k \geq 0$*) if and only if (*$\sum_{i=1}^{j} s_i \geq 0$ *for* $j = 1, 2, \ldots, k$*).*

Proof: Let $d_i = t_i - t_{i+1}$ for $i = 1, 2, \ldots, k-1$, and $d_k = t_k$. Then ($s_1t_1 + s_2t_2 + \ldots + s_kt_k \geq 0$ for any $t_1 \geq t_2 \geq \ldots \geq t_k \geq 0$) is equivalent to (($\sum_{i=1}^{1} s_i)d_1 + (\sum_{i=1}^{2} s_i)d_2 + \ldots + (\sum_{i=1}^{k} s_i)d_k \geq 0$ for any set of arbitrary nonnegative d_i). When

$\sum_{i=1}^{j} s_i \geq 0$ for $j = 1, 2, \ldots, k$, the above inequality is obviously true. If for some j, $\sum_{i=1}^{j} s_i < 0$, if we set $d_j > 0$ and $d_i = 0$ for all $i \neq j$, then the above inequality becomes false. So $\sum_{i=1}^{j} s_i \geq 0$ for $j = 1, 2, \ldots, k$ is both necessary and sufficient. ■

We are now ready to present the proof of Claim 2.

Proof: The individual rationality constraint can be written as $z_n = c_0 + c_1\hat{v}_1 + c_2\hat{v}_2 + c_3\hat{v}_3 + \ldots + c_{n-2}\hat{v}_{n-2} + c_{n-1}\hat{v}_{n-1} \geq 0$ for any bid vector $\hat{v}_1 \geq \hat{v}_2 \geq \ldots \geq \hat{v}_{n-1} \geq \hat{v}_n \geq 0$. We have already shown that $c_i = 0$ for $i \leq m$. Thus, the above can be simplified to $z_n = c_{m+1}\hat{v}_{m+1} + c_{m+2}\hat{v}_{m+2} + \ldots + c_{n-2}\hat{v}_{n-2} + c_{n-1}\hat{v}_{n-1} \geq 0$ for any bid vector. By the above lemma, this is equivalent to $\sum_{i=m+1}^{j} c_i \geq 0$ for $j = m+1, \ldots, n-1$. ■

Claim 3 *The non-deficit constraint and the worst-case constraint can also be written as linear inequalities involving only the c_i and k.*

Proof: The non-deficit constraint requires that for any bid vector, $z_1 + z_2 + \ldots + z_n \leq m\hat{v}_{m+1}$, where $z_i = c_0 + c_1\hat{v}_1 + c_2\hat{v}_2 + \ldots + c_{i-1}\hat{v}_{i-1} + c_i\hat{v}_{i+1} + \ldots + c_{n-1}\hat{v}_n$ for $i = 1, 2, \ldots, n$. Because $c_i = 0$ for $i \leq m$, we can simplify this inequality to

$q_{m+1}\hat{v}_{m+1} + q_{m+2}\hat{v}_{m+2} + \ldots + q_n\hat{v}_n \geq 0$
$q_{m+1} = m - (n-m-1)c_{m+1}$
$q_i = -(i-1)c_{i-1} - (n-i)c_i$, for $i = m+2, \ldots, n-1$ (when $m+2 > n-1$, this set of equalities is empty)
$q_n = -(n-1)c_{n-1}$

By the above lemma, this is equivalent to $\sum_{i=m+1}^{j} q_i \geq 0$ for $j = m+1, \ldots, n$. So, we can simplify further as follows:

$q_{m+1} \geq 0 \iff (n-m-1)c_{m+1} \leq m$
$q_{m+1} + \ldots + q_{m+i} \geq 0 \iff n\sum_{j=m+1}^{j=m+i-1} c_j + (n-m-i)c_{m+i} \leq m$ for $i = 2, \ldots, n-m-1$
$q_{m+1} + \ldots + q_n \geq 0 \iff n\sum_{j=m+1}^{j=n-1} c_j \leq m$

So, the non-deficit constraint can be written as a set of linear inequalities involving only the c_i.

The worst-case constraint can be also written as a set of linear inequalities, by the following reasoning. The worst-case constraint requires that for any bid input $z_1 + z_2 + \ldots + z_n \geq km\hat{v}_{m+1}$, where $z_i = c_0 + c_1\hat{v}_1 + c_2\hat{v}_2 + \ldots + c_{i-1}\hat{v}_{i-1} + c_i\hat{v}_{i+1} + \ldots + c_{n-1}\hat{v}_n$ for $i = 1, 2, \ldots, n$. Because $c_i = 0$ for $i \leq m$, we can simplify this inequality to

$Q_{m+1}\hat{v}_{m+1} + Q_{m+2}\hat{v}_{m+2} + \ldots + Q_n\hat{v}_n \geq 0$
$Q_{m+1} = (n-m-1)c_{m+1} - km$
$Q_i = (i-1)c_{i-1} + (n-i)c_i$, for $i = m+2, \ldots, n-1$

$Q_n = (n-1)c_{n-1}$

By the above lemma, this is equivalent to $\sum_{i=m+1}^{j} Q_i \geq 0$ for $j = m+1, \ldots, n$. So, we can simplify further as follows:

$Q_{m+1} \geq 0 \iff (n-m-1)c_{m+1} \geq km$
$Q_{m+1} + \ldots + Q_{m+i} \geq 0 \iff n\sum_{j=m+1}^{j=m+i-1} c_j + (n-m-i)c_{m+i} \geq km$ for $i = 2, \ldots, n-m-1$
$Q_{m+1} + \ldots + Q_n \geq 0 \iff n\sum_{j=m+1}^{j=n-1} c_j \geq km$

So, the worst-case constraint can also be written as a set of linear inequalities involving only the c_i and k. ■

Combining all the claims, we see that the original optimization problem can be transformed into the following linear program.

Variables: $c_{m+1}, c_{m+2}, \ldots, c_{n-1}, k$
Maximize k (the fraction redistributed in the worst case)
Subject to:
$\sum_{i=m+1}^{j} c_i \geq 0$ for $j = m+1, \ldots, n-1$
$km \leq (n-m-1)c_{m+1} \leq m$
$km \leq n\sum_{j=m+1}^{j=m+i-1} c_j + (n-m-i)c_{m+i} \leq m$ for $i = 2, \ldots, n-m-1$
$km \leq n\sum_{j=m+1}^{j=n-1} c_j \leq m$

6 Numerical Results

For selected values of n and m, we solved the linear program using Glpk (GNU Linear Programming Kit). In this section, we compare the resulting mechanisms with the Bailey-Cavallo mechanism.

6.1 Worst-case performance

In the table below, we present the results for a single unit ($m = 1$). The second column displays the fraction of the total VCG payment that is not redistributed in the worst case by the worst-case optimal mechanism—that is, it displays the value $1 - k$. (Displaying k would require too many significant digits.) Correspondingly, the third column displays the fraction of the total VCG payment that is not redistributed by the Bailey-Cavallo mechanism in the worst case (which is equal to $\frac{2}{n}$).

n	**Worst-Case Optimal Mechanism**	**Bailey-Cavallo Mechanism**
3	66.7%	66.7%
4	42.9%	50.0%
5	26.7%	40.0%
6	16.1%	33.3%
7	9.52%	28.6%
8	5.51%	25.0%
9	3.14%	22.2%
10	1.76%	20.0%
15	$8.55e-4$	13.3%
20	$3.62e-5$	10.0%
30	$5.40e-8$	$6.67e-2$
40	$7.09e-11$	$5.00e-2$

In the above table, we showed that when $m = 1$, the worst-case optimal mechanism significantly outperforms the Bailey-Cavallo mechanism in the worst case. For larger m ($m = 1, 2, 3, 4$, $n = m + 2, \ldots, 30$), we compare the worst-case performance of these two mechanisms in Figure 1. We see that for any m, when $n = m+2$, the worst-case optimal mechanism has the same worst-case performance as the Bailey-Cavallo mechanism (actually, in this case, the worst-case optimal mechanism is identical to the Bailey-Cavallo mechanism). When $n > m + 2$, the worst-case optimal mechanism outperforms the Bailey-Cavallo mechanism (in the worst case).

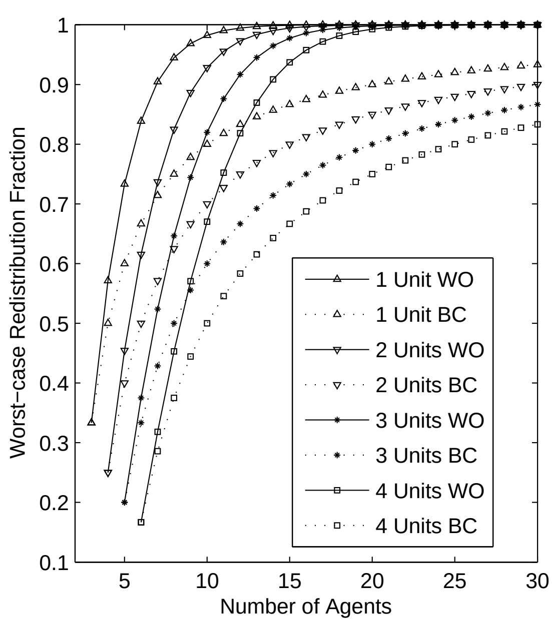

Figure 1: A comparison of the worst-case performance of the worst-case optimal mechanism (WO) and the Bailey-Cavallo mechanism (BC).

In Section 10, we will see that in the more general setting where agents have non-increasing marginal values, the worst-case redistribution fraction for the (generalized) worst-case optimal mechanism is the same as for the unit demand setting. The same is true for the Bailey-Cavallo mechanism. Hence, Figure 1 does not change in this more general setting.

6.2 Average-case performance

It is perhaps not surprising that the worst-case optimal mechanism significantly outperforms the Bailey-Cavallo mechanism in the worst case, because that is, after all, the case for which the former has been designed. We can also compare how much the mechanisms redistribute on average (say, when the bids are drawn i.i.d. from a uniform distribution over $[0, 1]$). In this case, the worst-case optimal mechanism does not always outperform the Bailey-Cavallo mechanism. The following table compares the expected amount of VCG payment that fails to be redistributed by the worst-case optimal mechanism and by the Bailey-Cavallo mechanism ($m = 1$).

n	**Worst-Case Optimal Mechanism**	**Bailey-Cavallo Mechanism**
3	0.1667	0.1667
4	0.1714	0.1000
5	0.08889	0.06667
6	0.06912	0.04762
7	0.03571	0.03571
8	0.02450	0.02778
9	0.01255	0.02222
10	0.008006	0.01818
15	$3.739e-4$	0.008333
20	$1.726e-5$	0.004762
30	$2.614e-8$	0.002151
40	$3.461e-11$	0.001220

We see that when n is small, the Bailey-Cavallo mechanism outperforms the worst-case optimal redistribution mechanism in expectation (except for the case $n = 3$, for which the two mechanisms are the same). When n is large ($n \geq 8$), the worst-case optimal redistribution mechanism outperforms the Bailey-Cavallo mechanism. The results are similar for larger m. That is, when n is small, the Bailey-Cavallo mechanism outperforms the worst-case optimal redistribution mechanism in expectation (except for the case $n = m + 2$, for which the two mechanisms are the same). When n is large (*e.g.* $n \geq 10$ for $m = 2$; $n \geq 13$ for $m = 3$; $n \geq 16$ for $m = 4$), the worst-case optimal redistribution mechanism performs better than the Bailey-Cavallo mechanism. In fact, this is not surprising: the expected amount that fails to be redistributed by the Bailey-Cavallo mechanism vanishes as $\Theta(\frac{1}{n^2})$. This is slower than the convergence rate of the *worst-case* redistribution fraction for the worst-case optimal mechanism (Corollary 1); and, of course, the average-case performance of the worst-case optimal mechanism must be at least as good as its worst-case performance. This also shows that the worst-case optimal mechanism asymptotically outperforms the Bailey-Cavallo mechanism, even in the average case.

6.3 A detailed example

Finally, let us present the result for the case $n = 5, m = 1$ in detail. By solving the above linear program, we find that the optimal values for the c_i are $c_2 = \frac{11}{45}, c_3 = -\frac{1}{9}$, and $c_4 = \frac{1}{15}$. That is, the redistribution payment received by each agent under the worst-case optimal mechanism is: $\frac{11}{45}$ times the second highest bid among the other agents, minus $\frac{1}{9}$ times the third highest bid among the other agents, plus $\frac{1}{15}$ times the fourth highest bid among the other agents.

agent a_1 receives $\frac{11}{45}\hat{v}_3 - \frac{1}{9}\hat{v}_4 + \frac{1}{15}\hat{v}_5$
agent a_2 receives $\frac{11}{45}\hat{v}_3 - \frac{1}{9}\hat{v}_4 + \frac{1}{15}\hat{v}_5$
agent a_3 receives $\frac{11}{45}\hat{v}_2 - \frac{1}{9}\hat{v}_4 + \frac{1}{15}\hat{v}_5$
agent a_4 receives $\frac{11}{45}\hat{v}_2 - \frac{1}{9}\hat{v}_3 + \frac{1}{15}\hat{v}_5$
agent a_5 receives $\frac{11}{45}\hat{v}_2 - \frac{1}{9}\hat{v}_3 + \frac{1}{15}\hat{v}_4$

The total amount redistributed by the worst-case optimal mechanism is $\frac{11}{15}\hat{v}_2 + \frac{4}{15}\hat{v}_3 - \frac{4}{15}\hat{v}_4 + \frac{4}{15}\hat{v}_5$; in the worst case, $\frac{11}{15}\hat{v}_2$ is redistributed. Hence, the fraction of the total VCG payment that is not redistributed is never more than $\frac{4}{15} = 26.7\%$.

As a specific example, for the bid vector $\hat{v}_1 = 4, \hat{v}_2 = 3, \hat{v}_3 = 2, \hat{v}_4 = 1, \hat{v}_5 = 1$, the total amount redistributed by the worst-case optimal redistribution mechanism is $\frac{11}{15}\hat{v}_2 + \frac{4}{15}\hat{v}_3 - \frac{4}{15}\hat{v}_4 + \frac{4}{15}\hat{v}_5 = \frac{11}{15}3 + \frac{4}{15}2 - \frac{4}{15}1 + \frac{4}{15}1 = \frac{41}{15}$. The total amount redistributed by the Bailey-Cavallo mechanism is $\frac{2}{5}\hat{v}_3 + \frac{3}{5}\hat{v}_2 = \frac{2}{5}2 + \frac{3}{5}3 = \frac{13}{5}$. Hence, for this bid vector, the worst-case optimal redistribution mechanism redistributes more.

As another specific example, for the bid vector $\hat{v}_1 = 4, \hat{v}_2 = 3, \hat{v}_3 = 2, \hat{v}_4 = 2, \hat{v}_5 = 1$, the total amount redistributed by the worst-case optimal redistribution mechanism is $\frac{11}{15}\hat{v}_2 + \frac{4}{15}\hat{v}_3 - \frac{4}{15}\hat{v}_4 + \frac{4}{15}\hat{v}_5 = \frac{11}{15}3 + \frac{4}{15}2 - \frac{4}{15}2 + \frac{4}{15}1 = \frac{37}{15}$. The total amount redistributed by the Bailey-Cavallo mechanism is still $\frac{13}{5}$. Hence, for this bid vector, the Bailey-Cavallo mechanism redistributes more.

7 Analytical Characterization of the Worst-Case Optimal Mechanism

We recall that our linear program has the following form:

Variables: $c_{m+1}, c_{m+2}, \ldots, c_{n-1}, k$
Maximize k (the fraction redistributed in the worst case)
Subject to:
$\sum_{i=m+1}^{j} c_i \geq 0$ for $j = m+1, \ldots, n-1$
$km \leq (n-m-1)c_{m+1} \leq m$
$km \leq n\sum_{j=m+1}^{j=m+i-1} c_j + (n-m-i)c_{m+i} \leq m$ for $i = 2, \ldots, n-m-1$
$km \leq n\sum_{j=m+1}^{j=n-1} c_j \leq m$

A linear program has no solution if and only if either the objective is unbounded, or the constraints are contradictory (there is no feasible solution). It is easy to see that k is bounded above by 1 (redistributing more than 100% violates the non-deficit constraint).

Also, a feasible solution always exists, for example, $k = 0$ and $c_i = 0$ for all i. So an optimal solution always exists. Observe that the linear program model depends only on the number of agents n and the number of units m. Hence the optimal solution is a function of n and m. It turns out that this optimal solution can be analytically characterized as follows.

Theorem 1 *For any m and n with $n \geq m + 2$, the worst-case optimal mechanism (among linear VCG redistribution mechanisms) is unique. For this mechanism, the fraction redistributed in the worst case is*

$$k^* = 1 - \frac{\binom{n-1}{m}}{\sum_{j=m}^{n-1} \binom{n-1}{j}}$$

The worst-case optimal mechanism is characterized by the following values for the c_i:

$$c_i^* = \frac{(-1)^{i+m-1}(n-m)\binom{n-1}{m-1}}{i \sum_{j=m}^{n-1} \binom{n-1}{j}} \frac{1}{\binom{n-1}{i}} \sum_{j=i}^{n-1} \binom{n-1}{j}$$

for $i = m + 1, \ldots, n - 1$.

It should be noted that we have proved $c_i = 0$ for $i \leq m$ in Claim 1.

Proof: We first rewrite the linear program as follows. We introduce new variables $x_{m+1}, x_{m+2}, \ldots, x_{n-1}$, defined by $x_j = \sum_{i=m+1}^{j} c_i$ for $j = m + 1, \ldots, n - 1$. The linear program then becomes:

Variables: $x_{m+1}, x_{m+2}, \ldots, x_{n-1}, k$
Maximize k
Subject to:
$km \leq (n - m - 1)x_{m+1} \leq m$
$km \leq (m + i)x_{m+i-1} + (n - m - i)x_{m+i} \leq m$ for $i = 2, \ldots, n - m - 1$
$km \leq nx_{n-1} \leq m$
$x_i \geq 0$ for $i = m + 1, m + 2, \ldots, n - 1$

We will prove that for any optimal solution to this linear program, $k = k^*$. Moreover, we will prove that when $k = k^*$, $x_j = \sum_{i=m+1}^{j} c_i^*$ for $j = m + 1, \ldots, n - 1$. This will prove the theorem.

We first make the following observations:

$$\begin{aligned}
&(n - m - 1)c_{m+1}^* \\
&= (n - m - 1)\tfrac{(n-m)\binom{n-1}{m-1}}{(m+1)\sum_{j=m}^{n-1}\binom{n-1}{j}} \tfrac{1}{\binom{n-1}{m+1}} \textstyle\sum_{j=m+1}^{n-1} \binom{n-1}{j} \\
&= (n - m - 1)\tfrac{(n-m)\binom{n-1}{m-1}}{(m+1)\sum_{j=m}^{n-1}\binom{n-1}{j}} \tfrac{1}{\binom{n-1}{m+1}} (\textstyle\sum_{j=m}^{n-1} \binom{n-1}{j} - \binom{n-1}{m}) \\
&= (n - m - 1)\tfrac{m}{n-m-1} - (n - m - 1)\tfrac{m\binom{n-1}{m}}{(n-m-1)\sum_{j=m}^{n-1}\binom{n-1}{j}} \\
&= m - (1 - k^*)m = k^*m
\end{aligned}$$

For $i = m+1, \ldots, n-2$,
$ic_i^* + (n-i-1)c_{i+1}^*$
$= i\frac{(-1)^{i+m-1}(n-m)\binom{n-1}{m-1}}{i\sum_{j=m}^{n-1}\binom{n-1}{j}}\frac{1}{\binom{n-1}{i}}\sum_{j=i}^{n-1}\binom{n-1}{j}$
$+ (n-i-1)\frac{(-1)^{i+m}(n-m)\binom{n-1}{m-1}}{(i+1)\sum_{j=m}^{n-1}\binom{n-1}{j}}\frac{1}{\binom{n-1}{i+1}}\sum_{j=i+1}^{n-1}\binom{n-1}{j}$
$= \frac{(-1)^{i+m-1}(n-m)\binom{n-1}{m-1}}{\sum_{j=m}^{n-1}\binom{n-1}{j}}\frac{1}{\binom{n-1}{i}}\sum_{j=i}^{n-1}\binom{n-1}{j}$
$- (n-i-1)\frac{(-1)^{i+m-1}(n-m)\binom{n-1}{m-1}}{(i+1)\sum_{j=m}^{n-1}\binom{n-1}{j}}\frac{i+1}{\binom{n-1}{i}(n-i-1)}\sum_{j=i+1}^{n-1}\binom{n-1}{j}$
$= \frac{(-1)^{i+m-1}(n-m)\binom{n-1}{m-1}}{\sum_{j=m}^{n-1}\binom{n-1}{j}}$
$= (-1)^{i+m-1}m(1-k^*)$

Finally,
$(n-1)c_{n-1}^*$
$= (n-1)\frac{(-1)^{n+m}(n-m)\binom{n-1}{m-1}}{(n-1)\sum_{j=m}^{n-1}\binom{n-1}{j}}\frac{1}{\binom{n-1}{n-1}}\sum_{j=n-1}^{n-1}\binom{n-1}{j}$
$= (-1)^{m+n}m(1-k^*)$

Summarizing the above, we have:
$(n-m-1)c_{m+1}^* = k^*m$
$(m+1)c_{m+1}^* + (n-m-2)c_{m+2}^* = m(1-k^*)$
$(m+2)c_{m+2}^* + (n-m-3)c_{m+3}^* = -m(1-k^*)$
$(m+3)c_{m+3}^* + (n-m-4)c_{m+4}^* = m(1-k^*)$
$\vdots$
$(n-3)c_{n-3}^* + 2c_{n-2}^* = (-1)^{m+n-2}m(1-k^*)$
$(n-2)c_{n-2}^* + c_{n-1}^* = (-1)^{m+n-1}m(1-k^*)$
$(n-1)c_{n-1}^* = (-1)^{m+n}m(1-k^*)$

Let $x_j^* = \sum_{i=m+1}^{j} c_i^*$ for $j = m+1, m+2, \ldots, n-1$, the first equation in the above tells us that $(n-m-1)x_{m+1}^* = k^*m$.

By adding the first two equations, we get $(m+2)x_{m+1}^* + (n-m-2)x_{m+2}^* = m$.

By adding the first three equations, we get $(m+3)x_{m+2}^* + (n-m-3)x_{m+3}^* = k^*m$.

By adding the first i equations, where $i = 2, \ldots, n-m-1$, we get $(m+i)x_{m+i-1}^* + (n-m-i)x_{m+i}^* = m$ if i is even; $(m+i)x_{m+i-1}^* + (n-m-i)x_{m+i}^* = k^*m$ if i is odd.

Finally by adding all the equations, we get $nx_{n-1}^* = m$ if $n-m$ is even; $nx_{n-1}^* = k^*m$ if $n-m$ is odd.

Thus, for all of the constraints other than the nonnegativity constraints, we have shown that they are satisfied by setting $x_j = x_j^* = \sum_{i=m+1}^{j} c_i^*$ and $k = k^*$. We next show that the nonnegativity constraints are satisfied by these settings as well.

For $m+1 \le i, i+1 \le n-1$, we have $\frac{1}{i}\frac{\sum_{j=i}^{n-1}\binom{n-1}{j}}{\binom{n-1}{i}} = \frac{1}{i}\sum_{j=i}^{n-1}\frac{i!(n-1-i)!}{j!(n-1-j)!} \ge \frac{1}{i+1}\sum_{j=i}^{n-2}\frac{i!(n-1-i)!}{j!(n-1-j)!} \ge \frac{1}{i+1}\sum_{j=i}^{n-2}\frac{(i+1)!(n-1-i-1)!}{(j+1)!(n-1-j-1)!} = \frac{1}{i+1}\frac{\sum_{j=i+1}^{n-1}\binom{n-1}{j}}{\binom{n-1}{i+1}}$

This implies that the absolute value of c_i^* is decreasing as i increases (if the c_i^* contains more than one number). We further observe that the sign of c_i^* alternates, with the first element c_{m+1}^* positive. So $x_j^* = \sum_{i=m+1}^{j} c_i^* \ge 0$ for all j. Thus, we have shown that these $x_i = x_i^*$ together with $k = k^*$ form a feasible solution of the linear program. We proceed to show that it is in fact the unique optimal solution.

First we prove the following claim:

Claim 4 *If $\hat{k}, \hat{x}_i$, $i = m+1, m+2, \ldots, n-1$ satisfy the following inequalities:*

$$\hat{k}m \le (n-m-1)\hat{x}_{m+1} \le m$$
$$\hat{k}m \le (m+i)\hat{x}_{m+i-1} + (n-m-i)\hat{x}_{m+i} \le m \text{ for } i = 2, \ldots, n-m-1$$
$$\hat{k}m \le n\hat{x}_{n-1} \le m$$
$$\hat{k} \ge k^*$$

then we must have that $\hat{x}_i = x_i^*$ and $\hat{k} = k^*$.

PROOF OF CLAIM. Consider the first inequality. We know that $(n-m-1)x_{m+1}^* = k^*m$, so $(n-m-1)\hat{x}_{m+1} \ge \hat{k}m \ge k^*m = (n-m-1)x_{m+1}^*$. It follows that $\hat{x}_{m+1} \ge x_{m+1}^*$ ($n-m-1 \ne 0$).

Now, consider the next inequality for $i=2$. We know that $(m+2)x_{m+1}^* + (n-m-2)x_{m+2}^* = m$. It follows that $(n-m-2)\hat{x}_{m+2} \le m-(m+2)\hat{x}_{m+1} \le m-(m+2)x_{m+1}^* = (n-m-2)x_{m+2}^*$, so $\hat{x}_{m+2} \le x_{m+2}^*$ ($i = 2 \le n-m-1 \Rightarrow n-m-2 \ne 0$).

Now consider the next inequality for $i=3$. We know that $(m+3)x_{m+2}^* + (n-m-3)x_{m+3}^* = m$. It follows that $(n-m-3)\hat{x}_{m+3} \ge \hat{k}m-(m+3)\hat{x}_{m+2} \ge k^*m-(m+3)x_{m+2}^* = (n-m-3)x_{m+3}^*$, so $\hat{x}_{m+3} \ge x_{m+3}^*$ ($i = 3 \le n-m-1 \Rightarrow n-m-3 \ne 0$).

Proceeding like this all the way up to $i = n-m-1$, we get that $\hat{x}_{m+i} \ge x_{m+i}^*$ if i is odd and $\hat{x}_{m+i} \le x_{m+i}^*$ if i is even. Moreover, if one inequality is strict, then all subsequent inequalities are strict. Now, if we can prove $\hat{x}_{n-1} = x_{n-1}^*$, it would follow that the x_i^* are equal to the $\hat{x}_i$ (which also implies that $\hat{k} = k^*$).

We consider two cases:

Case 1: $n-m$ is even. We have: $n-m$ even $\Rightarrow$ $n-m-1$ odd $\Rightarrow \hat{x}_{n-1} \ge x_{n-1}^*$. We also have: $n-m$ even $\Rightarrow nx_{n-1}^* = m$. Combining these two, we get $m = nx_{n-1}^* \le n\hat{x}_{n-1} \le m \Rightarrow \hat{x}_{n-1} = x_{n-1}^*$.

Case 2: $n-m$ is odd. In this case, we have $\hat{x}_{n-1} \le x_{n-1}^*$, and $nx_{n-1}^* = k^*m$. Then, we have: $k^*m \le \hat{k}m \le n\hat{x}_{n-1} \le nx_{n-1}^* = k^*m \Rightarrow \hat{x}_{n-1} = x_{n-1}^*$.

This completes the proof of the claim. □

It follows that if $\hat{k}, \hat{x}_i$, $i = m+1, m+2, \ldots, n-1$ is a feasible solution and $\hat{k} \ge k^*$, then since all the inequalities in Claim 4 are satisfied, we must have $\hat{x}_i = x_i^*$ and $\hat{k} = k^*$. Hence no other feasible solution is as good as the one described in the theorem. ■

Knowing the analytical characterization of the worst-case optimal mechanism provides us with at least two major benefits. First, using these formulas is computationally more efficient than solving the linear program using a general-purpose solver. Second, we can derive the following corollary.

Corollary 1 *If the number of units m is fixed, then as the number of agents n increases, the worst-case fraction redistributed linearly converges to 1, with a rate of convergence $\frac{1}{2}$. (That is, $\lim_{n\to\infty} \frac{1-k^*_{n+1}}{1-k^*_n} = \frac{1}{2}$. That is, in the limit, the fraction that is not redistributed halves for every additional agent.)*

We note that this is consistent with the experimental data for the single-unit case, where the worst-case remaining fraction roughly halves each time we add another agent. The worst-case fraction that is redistributed under the Bailey-Cavallo mechanism also converges to 1 as the number of agents goes to infinity, but the convergence is much slower—it does not converge linearly (that is, letting k_n^C be the fraction redistributed by the Bailey-Cavallo mechanism in the worst case for n agents, $\lim_{n\to\infty} \frac{1-k^C_{n+1}}{1-k^C_n} = lim_{n\to\infty} \frac{n}{n+1} = 1$). We now present the proof of the corollary.

Proof: When the number of agents is n, the worst-case fraction redistributed is $k^*_n = 1 - \frac{\binom{n-1}{m}}{\sum_{j=m}^{n-1}\binom{n-1}{j}}$. When the number of agents is $n+1$, the fraction becomes $k^*_{n+1} = 1 - \frac{\binom{n}{m}}{\sum_{j=m}^{n}\binom{n}{j}}$. For n sufficiently large, we will have $2^n - mn^{m-1} > 0$, and hence $\frac{1-k^*_{n+1}}{1-k^*_n} = \frac{\binom{n}{m}\sum_{j=m}^{n-1}\binom{n-1}{j}}{\binom{n-1}{m}\sum_{j=m}^{n}\binom{n}{j}} = \frac{n}{n-m}\frac{2^{n-1}-\sum_{j=0}^{m-1}\binom{n-1}{j}}{2^n-\sum_{j=0}^{m-1}\binom{n}{j}}$, and $\frac{n}{n-m}\frac{2^{n-1}-m(n-1)^{m-1}}{2^n} \le \frac{1-k^*_{n+1}}{1-k^*_n} \le \frac{n}{n-m}\frac{2^{n-1}}{2^n-mn^{m-1}}$ (because $\binom{n}{j} \le n^i$ if $j \le i$).

Since we have $\lim_{n\to\infty} \frac{n}{n-m}\frac{2^{n-1}-m(n-1)^{m-1}}{2^n} = \frac{1}{2}$, and $\lim_{n\to\infty} \frac{n}{n-m}\frac{2^{n-1}}{2^n-mn^{m-1}} = \frac{1}{2}$, it follows that $\lim_{n\to\infty} \frac{1-k^*_{n+1}}{1-k^*_n} = \frac{1}{2}$. ∎

8 Worst-Case Optimality Outside the Family

In this section, we prove that the worst-case optimal redistribution mechanism among linear VCG redistribution mechanisms is in fact optimal (in the worst case) among *all* redistribution mechanisms that are deterministic, anonymous, strategy-proof, efficient and satisfy the non-deficit constraint. Thus, restricting our attention to linear VCG redistribution mechanisms did not come at a loss.

To prove this theorem, we need the following lemma. This lemma is not new: it was informally stated by Cavallo [4]. For completeness, we present it here with a detailed proof.

Lemma 2 *A VCG redistribution mechanism is deterministic, anonymous and strategy-proof if and only if there exists a function $f : \mathbf{R^{n-1}} \to \mathbf{R}$, so that the redistribution payment z_i received by a_i satisfies*

$$z_i = f(\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{i-1}, \hat{v}_{i+1}, \ldots, \hat{v}_n)$$

for all i and all bid vectors.

Proof: First, let us prove the "only if" direction, that is, if a VCG redistribution mechanism is deterministic, anonymous and strategy-proof then there exists a deterministic function $f : \mathbf{R^{n-1}} \rightarrow \mathbf{R}$, which makes $z_i = f(\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{i-1}, \hat{v}_{i+1}, \ldots, \hat{v}_n)$ for all i and all bid vectors.

If a VCG redistribution mechanism is deterministic and anonymous, then for any bid vector $\hat{v}_1 \geq \hat{v}_2 \geq \ldots \geq \hat{v}_n$, the mechanism outputs a unique redistribution payment list: $z_1, z_2, \ldots, z_n$. Let $G : \mathbf{R^n} \rightarrow \mathbf{R^n}$ be the function that maps $\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n$ to $z_1, z_2, \ldots, z_n$ for all bid vectors. Let $H(i, x_1, x_2, \ldots, x_n)$ be the ith element of $G(x_1, x_2, \ldots, x_n)$, so that $z_i = H(i, \hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n)$ for all bid vectors and all $1 \leq i \leq n$. Because the mechanism is anonymous, two agents should receive the same redistribution payment if their bids are the same. So, if $\hat{v}_i = \hat{v}_j$, $H(i, \hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n) = H(j, \hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n)$. Hence, if we let $j = min\{t|\hat{v}_t = \hat{v}_i\}$, then $H(i, \hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n) = H(j, \hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n)$.

Let us define $K : \mathbf{R^n} \rightarrow \mathbf{N} \times \mathbf{R^n}$ as follows: $K(y, x_1, x_2, \ldots, x_{n-1}) = [j, w_1, w_2, \ldots, w_n]$, where $w_1, w_2, \ldots, w_n$ are $y, x_1, x_2, \ldots, x_{n-1}$ sorted in descending order, and $j = min\{t|w_t = y\}$. ($\{t|w_t = y\} \neq \emptyset$ because $y \in \{w_1, w_2, \ldots, w_n\}$).

Also let us define $F : \mathbf{R^n} \rightarrow \mathbf{R}$ by $F(\hat{v}_i, \hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{i-1}, \hat{v}_{i+1}, \ldots, \hat{v}_n) = H \circ K(\hat{v}_i, \hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{i-1}, \hat{v}_{i+1}, \ldots, \hat{v}_n) = H(min\{t|\hat{v}_t = \hat{v}_i\}, \hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n) = H(i, \hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n) = z_i$. That is, F is the redistribution payment to an agent that bids $\hat{v}_i$ when the other bids are $\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{i-1}, \hat{v}_{i+1}, \ldots, \hat{v}_n$.

Since our mechanism is required to be strategy-proof, and the space of valuations is unrestricted, z_i should be independent of $\hat{v}_i$ by Lemma 1 in Cavallo [4]. Hence, we can simply ignore the first variable input to F; let $f(x_1, x_2, \ldots, x_{n-1}) = F(0, x_1, x_2, \ldots, x_{n-1})$. So, we have $z_i = f(\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{i-1}, \hat{v}_{i+1}, \ldots, \hat{v}_n)$ for all bid vectors and i. This completes the proof for the "only if" direction.

For the "if" direction, if the redistribution payment received by a_i satisfies $z_i = f(\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{i-1}, \hat{v}_{i+1}, \ldots, \hat{v}_n)$ for all bid vectors and i, then this is clearly a deterministic and anonymous mechanism. To prove strategy-proofness, we observe that because an agent's redistribution payment is not affected by her own bid, her incentives are the same as in the VCG mechanism, which is strategy-proof. ∎

Now we are ready to introduce the next theorem:

Theorem 2 *For any m and n with $n \geq m + 2$, the worst-case optimal mechanism among the family of linear VCG redistribution mechanisms is worst-case optimal among all mechanisms that are deterministic, anonymous, strategy-proof, efficient and satisfy the non-deficit constraint.*

While we needed individual rationality earlier in the paper, this theorem does not mention it, that is, we cannot find a mechanism with better worst-case performance even if we sacrifice individual rationality. (The worst-case optimal linear VCG redistribution mechanism is of course individually rational.)

Proof: Suppose there is a redistribution mechanism (when the number of units is m and the number of agents is n) that satisfies all of the above properties and has a better worst-case performance than the worst-case optimal linear VCG redistribution mechanism, that is, its worst-case redistribution fraction $\hat{k}$ is strictly greater than k^*.

By Lemma 2, for this mechanism, there is a function $f : \mathbf{R^{n-1}} \to \mathbf{R}$ so that $z_i = f(\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{i-1}, \hat{v}_{i+1}, \ldots, \hat{v}_n)$ for all i and all bid vectors. We first prove that f has the following properties.

Claim 5 *$f(1, 1, \ldots, 1, 0, 0, \ldots, 0) = 0$ if the number of 1s is less than or equal to m.*

PROOF OF CLAIM. We assumed that for this mechanism, the worst-case redistribution fraction satisfies $\hat{k} > k^* \geq 0$. If the total VCG payment is x, the total redistribution payment should be in $[\hat{k}x, x]$ (non-deficit criterion). Consider the case where all agents bid 0, so that the total VCG payment is also 0. Hence, the total redistribution payment should be in $[\hat{k} \cdot 0, 0]$—that is, it should be 0. Hence every agent's redistribution payment $f(0, 0, \ldots, 0)$ must be 0.

Now, let $t_i = f(1, 1, \ldots, 1, 0, 0, \ldots, 0)$ where the number of 1s equals i. We proved $t_0 = 0$. If $t_{n-1} = 0$, consider the bid vector where everyone bids 1. The total VCG payment is m and the total redistribution payment is $nf(1, 1, \ldots, 1) = nt_{n-1} = 0$. This corresponds to 0% redistribution, which is contrary to our assumption that $\hat{k} > k^* \geq 0$. Now, consider $j = min\{i|t_i \neq 0\}$ (which is well-defined because $t_{n-1} \neq 0$). If $j > m$, the property is satisfied. If $j \leq m$, consider the bid vector where $\hat{v}_i = 1$ for $i \leq j$ and $\hat{v}_i = 0$ for all other i. Under this bid vector, the first j agents each get redistribution payment $t_{j-1} = 0$, and the remaining $n - j$ agents each get t_j. Thus, the total redistribution payment is $(n-j)t_j$. Because the total VCG payment for this bid vector is 0, we must have $(n - j)t_j = 0$. So $t_j = 0$ ($j \leq m < n$). But this is contrary to the definition of j. Hence $f(1, 1, \ldots, 1, 0, 0, \ldots, 0) = 0$ if the number of 1s is less than or equal to m. □

Claim 6 *f satisfies the following inequalities:*

$$\hat{k}m \leq (n - m - 1)t_{m+1} \leq m$$
$$\hat{k}m \leq (m + i)t_{m+i-1} + (n - m - i)t_{m+i} \leq m \text{ for } i = 2, 3, \ldots, n - m - 1$$
$$\hat{k}m \leq nt_{n-1} \leq m$$

Here t_i is defined as in the proof of Claim 5.

PROOF OF CLAIM. For $j = m + 1, \ldots, n$, consider the bid vectors where $\hat{v}_i = 1$ for $i \leq j$ and $\hat{v}_i = 0$ for all other i. These bid vectors together with the non-deficit constraint and worst-case constraint produce the above set of inequalities: for example, when $j = m + 1$, we consider the bid vector $\hat{v}_i = 1$ for $i \leq m + 1$ and $\hat{v}_i = 0$ for all other i. The first $m+1$ agents each receive a redistribution payment of $t_m = 0$, and all other agents each receive t_{m+1}. Thus, the total VCG redistribution is $(n-m-1)t_{m+1}$. The non-deficit constraint gives $(n-m-1)t_{m+1} \leq m$ (because the total VCG payment is m). The worst-case constraint gives $(n - m - 1)t_{m+1} \geq \hat{k}m$. Combining these two, we get the first inequality. The other inequalities can be obtained in the same way. □

We now observe that the inequalities in Claim 6, together with $\hat{k} \geq k^*$, are the same as those in Claim 4 (where the t_i are replaced by the $\hat{x}_i$). Thus, we can conclude that $\hat{k} = k^*$, which is contrary to our assumption $\hat{k} > k^*$. Hence no mechanism satisfying all the listed properties has a redistribution fraction greater than k^* in the worst case. ■

So far we have only talked about the case where $n \geq m + 2$. For the purpose of completeness, we provide the following claim for the $n = m + 1$ case. (We assume $n > m$ in the unit demand setting.)

Claim 7 *For any m and n with $n = m + 1$, the original VCG mechanism (that is, redistributing nothing) is (uniquely) worst-case optimal among all redistribution mechanisms that are deterministic, anonymous, strategy-proof, efficient and satisfy the non-deficit constraint.*

We recall that when $n = m + 1$, Claim 1 tells us that the only mechanism inside the family of linear redistribution mechanisms is the original VCG mechanism, so that this mechanism is automatically worst-case optimal inside this family. However, to prove the above claim, we need to show that it is worst-case optimal among *all* redistribution mechanisms that have the desired properties.

Proof: Suppose a redistribution mechanism exists that satisfies all of the above properties and has a worst-case performance as good as the original VCG mechanism, that is, its worst-case redistribution fraction is greater than or equal to 0. This implies that the total redistribution payment of this mechanism is always nonnegative.

By Lemma 2, for this mechanism, there is a function $f : \mathbf{R^{n-1}} \rightarrow \mathbf{R}$ so that $z_i = f(\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{i-1}, \hat{v}_{i+1}, \ldots, \hat{v}_n)$ for all i and all bid vectors. We will prove that $f(x_1, x_2, \ldots, x_{n-1}) = 0$ for all $x_1 \geq x_2 \geq \ldots \geq x_{n-1} \geq 0$.

First, consider the bid vector where $\hat{v}_i = 0$ for all i. Here, each agent receives a redistribution payment $f(0, 0, \ldots, 0)$. The total redistribution payment is then $nf(0, 0, \ldots, 0)$, which should be both greater than or equal to 0 (by the above observation) as well less than or equal to 0 (using the non-deficit criterion and the fact that the total VCG payment is 0). It follows that $f(0, 0, \ldots, 0) = 0$. Now, let us consider the bid vector where $\hat{v}_1 = x_1 \geq 0$ and $\hat{v}_i = 0$ for all other i. For this bid vector, the agent with the highest bid receives a redistribution payment of $f(0, 0, \ldots, 0) = 0$, and the other $n - 1$ agents each receive $f(x_1, 0, \ldots, 0)$. By the same reasoning as above, the total redistribution payment should be both greater than or equal to 0 and less than or equal to 0, hence $f(x_1, 0, \ldots, 0) = 0$ for all $x_1 \geq 0$.

Proceeding by induction, let us assume $f(x_1, x_2, \ldots, x_k, 0, \ldots, 0) = 0$ for all $x_1 \geq x_2 \geq \ldots \geq x_k \geq 0$, for some $k < n - 1$. Consider the bid vector where $\hat{v}_i = x_i$ for $i \leq k + 1$, and $\hat{v}_i = 0$ for all other i, where the x_i are arbitrary numbers satisfying $x_1 \geq x_2 \geq \ldots \geq x_k \geq x_{k+1} \geq 0$. For the agents with the highest $k + 1$ bids, their redistribution payment is specified by f acting on an input with only k non-zero variables. Hence they all receive 0 by induction assumption. The other $n - k - 1$ agents each receive $f(x_1, x_2, \ldots, x_k, x_{k+1}, 0, \ldots, 0)$. The total redistribution payment is then $(n - k - 1)f(x_1, x_2, \ldots, x_k, x_{k+1}, 0, \ldots, 0)$, which should be both greater than or equal to 0, and less than or equal to the total VCG payment. Now, in this bid vector, the lowest bid is 0 because $k + 1 < n$. But since $n = m + 1$, the total VCG payment is $m\hat{v}_n = 0$. So we have $f(x_1, x_2, \ldots, x_k, x_{k+1}, 0, \ldots, 0) = 0$ for all $x_1 \geq x_2 \geq \ldots \geq x_k \geq x_{k+1} \geq 0$. By induction, this statement holds for all

$k < n-1$; when $k+1 = n-1$, we have $f(x_1, x_2, \ldots, x_{n-2}, x_{n-1}) = 0$ for all $x_1 \geq x_2 \geq \ldots \geq x_{n-2} \geq x_{n-1} \geq 0$. Hence, in this mechanism, the redistribution payment is always 0; that is, the mechanism is just the original VCG mechanism. ■

Incidentally, we obtain the following corollary:

Corollary 2 *No VCG redistribution mechanism satisfies all of the following: determinism, anonymity, strategy-proofness, efficiency, and (strong) budget balance. This holds for any* $n \geq m+1$.

Proof: *For the case* $n \geq m+2$*:* If such a mechanism exists, its worst-case performance would be better than that of the worst-case optimal linear VCG redistribution mechanism, which by Theorem 1 obtains a redistribution fraction strictly less than 1. But Theorem 2 shows that it is impossible to outperform this mechanism in the worst case.

For the case $n = m+1$*:* If such a mechanism exists, it would perform as well as the original VCG mechanism in the worst case, which implies that it is identical to the VCG mechanism by Claim 7. But the VCG mechanism is not (strongly) budget balanced. ■

9 Worst-Case Optimal Mechanism When Deficits Are Allowed

In the previous section, we showed that even if the individual rationality requirement is dropped, the worst-case optimal redistribution mechanism remains the same. In this section, we consider dropping the non-deficit requirement, and try to find the redistribution mechanism that deviates the least from budget balance (in the worst case).

We define the *imbalance* to be the absolute difference between the total redistribution and the total VCG payment, and define the *imbalance fraction* to be the ratio between the imbalance and the total VCG payment. Our goal is to minimize the worst-case imbalance fraction. Finding the optimal linear mechanism corresponds to the following optimization model:

Minimize k_d (the imbalance fraction in the worst case)
Subject to:
For every bid vector $\hat{v}_1 \geq \hat{v}_2 \geq \ldots \geq \hat{v}_n \geq 0$
$z_n \geq 0$ (individual rationality)
$|z_1 + z_2 + \ldots + z_n - m\hat{v}_{m+1}| \leq k_d m \hat{v}_{m+1}$ (imbalance constraint)
We recall that $z_i = c_0 + c_1\hat{v}_1 + c_2\hat{v}_2 + \ldots + c_{i-1}\hat{v}_{i-1} + c_i\hat{v}_{i+1} + \ldots + c_{n-1}\hat{v}_n$

The imbalance constraint can also be written as
$(1-k_d)m\hat{v}_{m+1} \leq z_1 + z_2 + \ldots + z_n \leq (1+k_d)m\hat{v}_{m+1}$
The above optimization model can be transformed into a linear program, based on the following observations.

Claim 8 *If $c_0, c_1, \ldots, c_{n-1}$ satisfy both the individual rationality and the imbalance constraints, then $c_i = 0$ for $i = 0, \ldots, m$.*

The proof is a slight modification of the proof of Claim 1.

Proof: First, let us prove that $c_0 = 0$. Consider the bid vector in which $\hat{v}_i = 0$ for all i. To obtain individual rationality, we must have $c_0 \geq 0$. To satisfy the imbalance constraint, we must have $c_0 \leq 0$. Thus we know $c_0 = 0$. Now, if $c_i = 0$ for all i, there is nothing to prove. Otherwise, let $j = min\{i | c_i \neq 0\}$. Assume that $j \leq m$. We recall that we can write the individual rationality constraint as follows: $z_n = c_0 + c_1\hat{v}_1 + c_2\hat{v}_2 + c_3\hat{v}_3 + \ldots + c_{n-2}\hat{v}_{n-2} + c_{n-1}\hat{v}_{n-1} \geq 0$ for any bid vector. Let us consider the bid vector in which $\hat{v}_i = 1$ for $i \leq j$ and $\hat{v}_i = 0$ for the rest. In this case $z_n = c_j$, so we must have $c_j \geq 0$. The imbalance constraint requires that : $z_1 + z_2 + \ldots + z_n \leq (1 + k_d)m\hat{v}_{m+1}$ for any bid vector. Consider the same bid vector as above. We have $z_i = 0$ for $i \leq j$, because for these bids, the jth highest other bid has value 0, so all the c_i that are nonzero are multiplied by 0. For $i > j$, we have $z_i = c_j$, because the jth highest other bid has value 1, and all lower bids have value 0. So the imbalance constraint tells us that $c_j(n-j) \leq (1+k_d)m\hat{v}_{m+1}$. Because $j \leq m$, $\hat{v}_{m+1} = 0$, so the right hand side is 0. We also have $n - j > 0$ because $j \leq m < n$. So $c_j \leq 0$. Because we have already established that $c_j \geq 0$, it follows that $c_j = 0$; but this is contrary to assumption. So $j > m$. ∎

Claim 9 *The imbalance constraint can be written as linear inequalities involving only the c_i and k_d.*

The proof is a slight modification of the proof of Claim 3.

Proof: The imbalance constraint requires that for any bid vector, $(1 - k_d)m\hat{v}_{m+1} \leq z_1 + z_2 + \ldots + z_n \leq (1 + k_d)m\hat{v}_{m+1}$, where $z_i = c_0 + c_1\hat{v}_1 + c_2\hat{v}_2 + \ldots + c_{i-1}\hat{v}_{i-1} + c_i\hat{v}_{i+1} + \ldots + c_{n-1}\hat{v}_n$ for $i = 1, 2, \ldots, n$. Because $c_i = 0$ for $i \leq m$, we can simplify this inequality to

$$q_{m+1}\hat{v}_{m+1} + q_{m+2}\hat{v}_{m+2} + \ldots + q_n\hat{v}_n \geq 0$$
$$q_{m+1} = (n - m - 1)c_{m+1} - (1 - k_d)m$$
$$q_i = (i-1)c_{i-1} + (n-i)c_i, \text{ for } i = m+2, \ldots, n-1$$
$$q_n = (n-1)c_{n-1}$$

$$Q_{m+1}\hat{v}_{m+1} + Q_{m+2}\hat{v}_{m+2} + \ldots + Q_n\hat{v}_n \leq 0$$
$$Q_{m+1} = (n - m - 1)c_{m+1} - (1 + k_d)m$$
$$Q_i = (i-1)c_{i-1} + (n-i)c_i, \text{ for } i = m+2, \ldots, n-1$$
$$Q_n = (n-1)c_{n-1}$$

By Lemma 1, this is equivalent to $\sum_{i=m+1}^{j} q_i \geq 0$ for $j = m+1, \ldots, n$ and $\sum_{i=m+1}^{j} Q_i \leq 0$ for $j = m+1, \ldots, n$. So, we can simplify further as follows:

$(1 - k_d)m \leq (n - m - 1)c_{m+1} \leq (1 + k_d)m$

$(1-k_d)m \leq n\sum_{j=m+1}^{j=m+i-1} c_j + (n-m-i)c_{m+i} \leq (1+k_d)m$ for $i = 2, \ldots, n - m - 1$

$(1-k_d)m \leq n\sum_{j=m+1}^{j=n-1} c_j \leq (1+k_d)m$

So, the imbalance constraint can also be written as a set of linear inequalities involving only the c_i and k_d. ■

Combining all the claims (together with Claim 2), we see that the original optimization problem can be transformed into the following linear program.

Variables: $c_{m+1}, c_{m+2}, \ldots, c_{n-1}, k_d$
Minimize k_d (the imbalance fraction in the worst case)
Subject to:
$\sum_{i=m+1}^{j} c_i \geq 0$ for $j = m+1, \ldots, n-1$
$(1-k_d)m \leq (n-m-1)c_{m+1} \leq (1+k_d)m$
$(1-k_d)m \leq n\sum_{j=m+1}^{j=m+i-1} c_j + (n-m-i)c_{m+i} \leq (1+k_d)m$ for $i = 2, \ldots, n-m-1$
$(1-k_d)m \leq n\sum_{j=m+1}^{j=n-1} c_j \leq (1+k_d)m$

For this model, it is easy to see that k_d is bounded below by 0. Also, $k_d = 1$ and $c_i = 0$ for all i form a feasible solution. So an optimal solution always exists. As in the case where deficits are not allowed, the optimal solution can be analytically characterized. The characterization is the following:

Theorem 3 *For any m and n with $n \geq m+2$, the worst-case optimal mechanism with deficits (among linear VCG redistribution mechanisms) is unique. For this mechanism, the imbalance fraction in the worst case is*

$$k_d^* = \frac{\binom{n-1}{m}}{\sum_{j=m+1}^{n}\binom{n}{j}}$$

The worst-case optimal mechanism with deficits is characterized by the following values for the c_i:

$$c_i^* = \frac{2(-1)^{i+m-1}(n-m)\binom{n-1}{m-1}}{i\sum_{j=m+1}^{n}\binom{n}{j}} \frac{1}{\binom{n-1}{i}} \sum_{j=i}^{n-1}\binom{n-1}{j}$$

for $i = m+1, \ldots, n-1$.

From Claim 8 it follows that $c_i = 0$ for $i \leq m$.

Proof: Let $\alpha = k_d^*/(1-k^*)$, where k^* is the worst-case optimal redistribution fraction in Theorem 1. To avoid ambiguity, we refer to the c_i^* in Theorem 1 as c_i^{w*}, and to the c_i^* here as c_i^{d*}. Inspection reveals that $c_i^{d*} = 2\alpha c_i^{w*}$ for all i. We have shown in Theorem 1 that
$\sum_{i=m+1}^{j} c_i^{w*} \geq 0$ for $j = m+1, \ldots, n-1$
$k^* m \leq (n-m-1)c_{m+1}^{w*} \leq m$
$k^* m \leq n\sum_{j=m+1}^{j=m+i-1} c_j^{w*} + (n-m-i)c_{m+i}^{w*} \leq m$ for $i = 2, \ldots, n-m-1$

$k^*m \le n\sum_{j=m+1}^{j=n-1} c_j^{w*} \le m$

So we have
$\sum_{i=m+1}^{j} c_i^{d*} \ge 0$ for $j = m+1, \dots, n-1$ (α is positive)
$2\alpha k^*m \le (n-m-1)c_{m+1}^{d*} \le 2\alpha m$
$2\alpha k^*m \le n\sum_{j=m+1}^{j=m+i-1} c_j^{d*} + (n-m-i)c_{m+i}^{d*} \le 2\alpha m$ for $i = 2, \dots, n-m-1$
$2\alpha k^*m \le n\sum_{j=m+1}^{j=n-1} c_j^{d*} \le 2\alpha m$

A sequence of algebraic manipulations reveals that $2\alpha k^* = (1-k_d^*)$ and $2\alpha = (1+k_d^*)$. Hence, k_d^* and the c_i^{d*} form a feasible solution, because we have
$\sum_{i=m+1}^{j} c_i^{d*} \ge 0$ for $j = m+1, \dots, n-1$
$(1-k_d^*)m \le (n-m-1)c_{m+1}^{d*} \le (1+k_d^*)m$
$(1-k_d^*)m \le n\sum_{j=m+1}^{j=m+i-1} c_j^{d*} + (n-m-i)c_{m+i}^{d*} \le (1+k_d^*)m$ for $i = 2, \dots, n-m-1$
$(1-k_d^*)m \le n\sum_{j=m+1}^{j=n-1} c_j^{d*} \le (1+k_d^*)m$

We proceed to show that it is in fact the unique optimal solution. Suppose $\hat{c}_i$ and $\hat{k}_d$ form a feasible solution, and $\hat{k}_d \le k_d^*$. We have
$(1-k_d^*)m \le (1-\hat{k}_d)m \le (n-m-1)\hat{c}_{m+1} \le (1+\hat{k}_d)m \le (1+k_d^*)m$
$(1-k_d^*)m \le (1-\hat{k}_d)m \le n\sum_{j=m+1}^{j=m+i-1} \hat{c}_j + (n-m-i)\hat{c}_{m+i} \le (1+\hat{k}_d)m \le (1+k_d^*)m$ for $i = 2, \dots, n-m-1$
$(1-k_d^*)m \le (1-\hat{k}_d)m \le n\sum_{j=m+1}^{j=n-1} \hat{c}_j \le (1+\hat{k}_d)m \le (1+k_d^*)m$

We introduce new variables $x_{m+1}, x_{m+2}, \dots, x_{n-1}$, defined by $x_j = \frac{1}{2\alpha}\sum_{i=m+1}^{j} \hat{c}_i$ for $j = m+1, \dots, n-1$. The above inequalities can be rewritten in terms of x_i, we have
$k^*m \le (n-m-1)x_{m+1} \le m$
$k^*m \le (m+i)x_{m+i-1} + (n-m-i)x_{m+i} \le m$ for $i = 2, \dots, n-m-1$
$k^*m \le nx_{n-1} \le m$

However, in Claim 4, we proved that these inequalities have a unique solution. Therefore, there is only one value that each of $\hat{c}_i$ and $\hat{k}_d$ can have. This proves that k_d^* and the c_i^{d*} form the unique optimal solution. ∎

$\alpha = k_d^*/(1-k^*)$ can be interpreted as the ratio between the imbalance fraction of the worst-case optimal mechanism with deficits (among linear VCG redistribution mechanisms) and the imbalance fraction of the worst-case optimal mechanism without deficits. This ratio can be expressed as follows:
$\alpha = k_d^*/(1-k^*) = \frac{\sum_{j=m}^{n-1}\binom{n-1}{j}}{\sum_{j=m+1}^{n}\binom{n}{j}} = \frac{\sum_{j=m+1}^{n}\binom{n-1}{j-1}}{\sum_{j=m+1}^{n}\binom{n}{j}} = \frac{\sum_{j=m+1}^{n}((j/n)\binom{n}{j})}{\sum_{j=m+1}^{n}\binom{n}{j}}$
For fixed n, this ratio increases as m increases. (This is because as we decrease m by 1, the ratio of the additional terms in the fraction decreases.) When $m = 1$, $\alpha = \frac{2^{n-1}-1}{2^n-n-1}$ (for large n, roughly $\frac{1}{2}$); when $m = n-2$, $\alpha = \frac{n}{n+1}$ (for large n, roughly 1). Hence, if m is small (relative to n), the worst-case optimal linear VCG

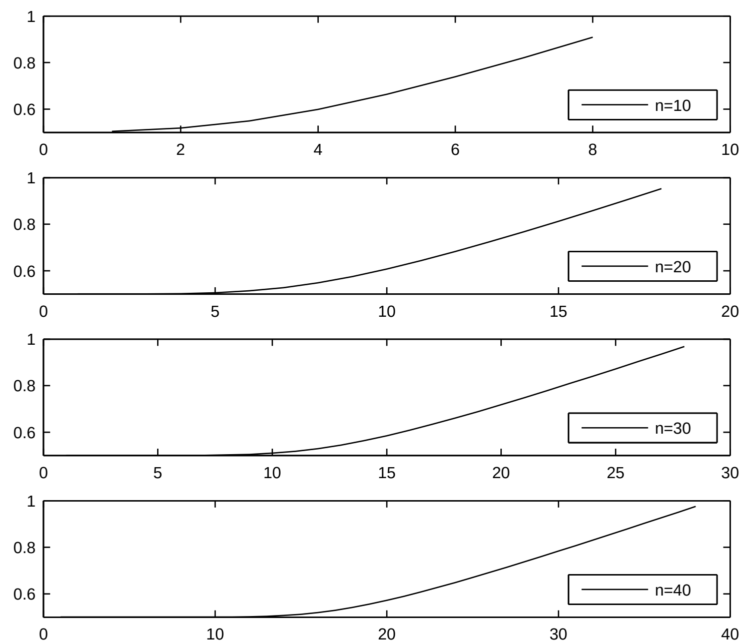

Figure 2: The value of α from $m = 1$ to $n - 2$

redistribution mechanism with deficits is much closer to budget balance than the worst-case optimal mechanism without deficits; if m is large (relative to n), they are about the same. On the other hand, when m is small relative to n, then the worst-case optimal redistribution fraction is large even with the non-deficit requirement. This means that the non-deficit constraint does not come at a great cost. Figure 2 shows how α changes as a function of m and n.

Now we prove that the worst-case optimal linear VCG redistribution mechanism with deficits is in fact optimal among *all* redistribution mechanisms that are deterministic, anonymous, strategy-proof and efficient.

Theorem 4 *For any m and n with $n \geq m + 2$, the worst-case optimal mechanism with deficits among linear VCG redistribution mechanisms has the smallest worst-case imbalance fraction among all VCG redistribution mechanisms that are deterministic, anonymous, strategy-proof and efficient.*

As in the case of Theorem 2, there is no redistribution mechanism with a smaller worst-case imbalance fraction even if we sacrifice individual rationality.

Proof: Suppose there is a redistribution mechanism (when the number of units is m and the number of agents is n) that satisfies all of the above properties and has a smaller worst-case imbalance fraction than that of the worst-case optimal linear VCG redistribution mechanism with deficits—that is, its worst-case imbalance fraction $\hat{k}_d$ is strictly less than k_d^*.

By Lemma 2, for this mechanism, there is a function $f : \mathbf{R^{n-1}} \rightarrow \mathbf{R}$ so that $z_i = f(\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{i-1}, \hat{v}_{i+1}, \ldots, \hat{v}_n)$ for all i and all bid vectors. The following properties of f follow from straightforward modifications of the proofs of Claim 5 and Claim 6.

Claim 10 *$f(1, 1, \ldots, 1, 0, 0, \ldots, 0) = 0$ if the number of 1s is less than or equal to m.*

Claim 11 *f satisfies the following inequalities:*

$$(1 - \hat{k}_d)m \le (n - m - 1)t_{m+1} \le (1 + \hat{k}_d)m$$
$$(1 - \hat{k}_d)m \le (m + i)t_{m+i-1} + (n - m - i)t_{m+i} \le (1 + \hat{k}_d)m \text{ for}$$
$$i = 2, 3, \ldots, n - m - 1$$
$$(1 - \hat{k}_d)m \le nt_{n-1} \le (1 + \hat{k}_d)m$$

$t_i = f(1, 1, \ldots, 1, 0, 0, \ldots, 0)$ where the number of 1s equals i

Let $x_i = \frac{1}{2\alpha}t_i$ for $i = m + 1, \ldots, n - 1$. Since $\hat{k}_d < k_d^*$, we have
$k^*m < \frac{1}{2\alpha}(1 - \hat{k}_d)m \le (n - m - 1)x_{m+1} \le \frac{1}{2\alpha}(1 + \hat{k}_d)m < m$
$k^*m < \frac{1}{2\alpha}(1 - \hat{k}_d)m \le (m + i)x_{m+i-1} + (n - m - i)x_{m+i} \le \frac{1}{2\alpha}(1 + \hat{k}_d)m < m$ for $i = 2, 3, \ldots, n - m - 1$
$k^*m < \frac{1}{2\alpha}(1 - \hat{k}_d)m \le nx_{n-1} \le \frac{1}{2\alpha}(1 + \hat{k}_d)m < m$

By Claim 4, the above system of inequalities cannot hold. Hence no mechanism satisfying all the listed properties has an imbalance fraction less than k_d^* in the worst case. ■

For the purpose of completeness, we note the following claim, which follows from a straightforward modification of the proof of Claim 7.

Claim 12 *For any m and n with $n = m + 1$, the original VCG mechanism (that is, redistributing nothing) is (uniquely) the worst-case optimal mechanism with deficits among all redistribution mechanisms that are deterministic, anonymous, strategy-proof and efficient.*

Proof: Suppose a redistribution mechanism exists that satisfies all of the above properties and has a worst-case performance as good as the original VCG mechanism, that is, its worst-case imbalance fraction is less than or equal to 100%.

By Lemma 2, for this mechanism, there is a function $f : \mathbf{R^{n-1}} \rightarrow \mathbf{R}$ so that $z_i = f(\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{i-1}, \hat{v}_{i+1}, \ldots, \hat{v}_n)$ for all i and all bid vectors. We will prove that $f(x_1, x_2, \ldots, x_{n-1}) = 0$ for all $x_1 \ge x_2 \ge \ldots \ge x_{n-1} \ge 0$.

First, consider the bid vector where $\hat{v}_i = 0$ for all i. Here, each agent receives a redistribution payment $f(0, 0, \ldots, 0)$. The total redistribution payment is then $nf(0, 0, \ldots, 0)$, which should be 0, because the total VCG payment is 0 (under 100% imbalance fraction, the imbalance is still 0). It follows that $f(0, 0, \ldots, 0) = 0$. Now, let us consider the bid vector where $\hat{v}_1 = x_1 \ge 0$ and $\hat{v}_i = 0$ for all other i. For this bid vector, the agent with the highest bid receives a redistribution payment of $f(0, 0, \ldots, 0) = 0$, and the other $n-1$ agents each receive $f(x_1, 0, \ldots, 0)$. By the same reasoning as above, the total redistribution payment should be 0, hence $f(x_1, 0, \ldots, 0) = 0$ for all $x_1 \geq 0$.

Proceeding by induction, let us assume $f(x_1, x_2, \ldots, x_k, 0, \ldots, 0) = 0$ for all $x_1 \geq x_2 \geq \ldots \geq x_k \geq 0$, for some $k < n-1$. Consider the bid vector where $\hat{v}_i = x_i$ for $i \leq k+1$, and $\hat{v}_i = 0$ for all other i, where the x_i are arbitrary numbers satisfying $x_1 \geq x_2 \geq \ldots \geq x_k \geq x_{k+1} \geq 0$. For the agents with the highest $k+1$ bids, their redistribution payment is specified by f acting on an input with only k non-zero variables. Hence they all receive 0 by induction assumption. The other $n-k-1$ agents each receive $f(x_1, x_2, \ldots, x_k, x_{k+1}, 0, \ldots, 0)$. The total redistribution payment is then $(n-k-1)f(x_1, x_2, \ldots, x_k, x_{k+1}, 0, \ldots, 0)$. Now, in this bid vector, the lowest bid is 0 because $k+1 < n$. But since $n = m+1$, the total VCG payment is $m\hat{v}_n = 0$, which forces the total redistribution payment to be 0. So we have $f(x_1, x_2, \ldots, x_k, x_{k+1}, 0, \ldots, 0) = 0$ for all $x_1 \geq x_2 \geq \ldots \geq x_k \geq x_{k+1} \geq 0$. By induction, this statement holds for all $k < n-1$; when $k+1 = n-1$, we have $f(x_1, x_2, \ldots, x_{n-2}, x_{n-1}) = 0$ for all $x_1 \geq x_2 \geq \ldots \geq x_{n-2} \geq x_{n-1} \geq 0$. Hence, in this mechanism, the redistribution payment is always 0; that is, the mechanism is just the original VCG mechanism. ∎

10 Multi-Unit Auction with Nonincreasing Marginal Values

In this section, we consider the more general setting where the agents have nonincreasing marginal values. (Units remain indistinguishable.) An agent's bid is now a vector of m elements, with the jth element denoting this agent's marginal value for getting her jth unit (and the elements are nonincreasing in j). That is, the agent's valuation for receiving j units is the sum of the first j elements. Let the set of agents be $\{a_1, a_2, \ldots, a_n\}$, where a_i is the agent with the ith highest initial marginal value (the marginal value for winning the first unit).

We still consider only the case where $m \leq n-2$, because if $m \geq n-1$, then the original VCG mechanism is worst-case optimal, both with and without deficits (we will show this in Claim 19).

The VCG mechanism requires us to find the efficient allocation. Because marginal values are nonincreasing, this can be achieved by the following greedy algorithm. At each step, we sort the agents according to their upcoming marginal values (their values for winning their next unit), and allocate one unit to the agent with the highest such value. We continue until there are no units left, or the remaining agents all have upcoming marginal values of zero (in this case, we simply throw away the remaining units). Given that marginal values are nonincreasing, the following greedy algorithm is effectively the same (in terms of the allocation process): sort *all* the marginal values (not just those for upcoming units), and accept them in decreasing order. Because marginal values are nonincreasing, when we accept one of them, this marginal value does in fact correspond to that agent's utility for receiving another unit at that point. In the proofs below, this greedy algorithm will provide a useful view of how units are allocated.

In the efficient allocation, only agents $a_1, \ldots, a_m$ can possibly win, and the VCG payments are determined by the bids of $a_1, \ldots, a_{m+1}$ (because when we remove an agent, only the top m remaining agents can possibly win).

We will generalize the worst-case optimal mechanism (both with and without deficits) to the current setting, and show in each case that the generalized mechanism has the same worst-case performance. This implies that there does not exist another redistribution mechanism with better worst-case performance (because such a mechanism would also have better worst-case performance in the more specific unit demand setting).

Let us use A to denote the set of all agents, and A_{-i} to denote the set of agents other than a_i. Because the mechanisms under consideration are strategy-proof, agents can be expected to report truthfully; hence, we do not make a sharp distinction between an agent and her bid. We define the following functions:

- $VCG : \mathcal{P}(A) \to \mathbf{R}$

  For any subset S of A, let $VCG(S)$ be the total VCG payment when only the agents in S participate in the auction.

- $E : \mathcal{P}(A) \to \mathbf{R}$

  For any subset S of A, let $E(S)$ be the total efficiency (that is, the total utility not taking payments into account) when only the agents in S participate in the auction.

- $e : \mathcal{P}(A) \times A \to \mathbf{R}$

  For any subset S of A and any $a \in S$, let $e(S, a)$ be the utility (not taking payments into account) of agent a, when only the agents in S participate in the auction. We note that $E(S) = \sum_{a \in S} e(S, a)$.

- $U : \mathcal{P}(A) \times \mathbf{N} \to \mathcal{P}(A)$

  For any subset S of A, any integer i ($1 \leq i \leq |S|$), let $U(S, i)$ be the set that results after removing the agent with the ith highest initial marginal value in S from S. (If there is a tie, this tie is broken according to the original order $a_1, \ldots, a_n$.)

- $R : \mathcal{P}(A) \times \mathbf{N} \to \mathbf{R}$

  For any subset S of A, any integer i ($0 \leq i \leq |S| - m$), let $R(S, i) = \frac{1}{m+i} \sum_{j=1}^{m+i} R(U(S, j), i-1)$ if $i > 0$, and $R(S, 0) = VCG(S)$. We emphasize that this is a recursive definition: for $i > 0$, $R(S, i)$ is obtained by computing, for each j with $1 \leq j \leq m + i$, $R(U(S, j), i - 1)$ (that is, the value of the function R after removing the jth agent in S from S, and decreasing i by one), and taking the average. For $i = 0$, it is simply the total VCG payment if only the agents from S are present. Shortly, we will prove some properties of this function that clarify its usefulness to our mechanism.

Let $V_i = R(A, i)$ for all i ($0 \leq i \leq n - m$). We first prove several claims.

Claim 13 *If we have $S, \hat{S} \in \mathcal{P}(A)$, $S \subseteq \hat{S}$, and $|\hat{S}| = |S| + 1$, then for any $a \in S$, we have $E(\hat{S}) - E(\hat{S} - \{a\}) \leq E(S) - E(S - \{a\})$. That is, E is submodular.*

Proof: Suppose a wins k units when only agents in $\hat{S}$ participate in the auction. We modify a's bid by setting a's marginal value for winning the $(k+1)$th unit to 0. This modification does not change the value of the left-hand side of the inequality, and it will never increase the value of the right-hand side of the inequality. Therefore, it suffices to prove that the inequality holds after the modification.

After the modification, a still wins exactly k units when only agents in S participate in the auction (as can be seen, for example, by considering the greedy allocation algorithm that we presented previously). Now, $E(\hat{S}) - E(\hat{S} - \{a\})$ is the increase in the total efficiency due to a winning k units (rather than other agents winning these units). That is, it equals the utility of a (not counting payments) minus the sum of the k upcoming marginal values in the greedy allocation algorithm—that is, the marginal values that rank $(m-k+1)$th to mth among marginal values extracted from the bids of the agents in $\hat{S} - \{a\}$. Similarly, $E(S) - E(S - \{a\})$ equals the utility of a minus the sum of the k upcoming marginal values—that is, the marginal values that rank $(m-k+1)$th to mth among marginal values extracted from the bids of the agents in $S - \{a\}$. But the k upcoming values in the second case must be smaller than those in the first case, because $S \subseteq \hat{S}$. Hence, $E(\hat{S}) - E(\hat{S} - \{a\}) \leq E(S) - E(S - \{a\})$. ■

The next claim shows that in the setting that we are considering, revenue is nondecreasing in agents. (This is not true in more general settings [29, 2, 7, 31, 32, 33].)

Claim 14 *For any $S, \hat{S} \in \mathcal{P}(A)$, if $S \subseteq \hat{S}$, then $VCG(S) \leq VCG(\hat{S})$. That is, revenue is nondecreasing in agents.*

Proof: We will prove the following equivalent statement instead: for any $S, \hat{S} \in \mathcal{P}(A)$, if $S \subseteq \hat{S}$ and $\hat{S}$ has exactly one more element than S, we have $VCG(S) \leq VCG(\hat{S})$.

Suppose $S = \{a'_1, a'_2, \ldots, a'_{|S|}\}$, where a'_i is the agent with the ith-highest initial marginal value in S. Since we know that only the agents from a'_1 to a'_m can possibly win any units, we have $VCG(S) = \sum_{i=1}^{m}(E(S - \{a'_i\}) - \sum_{j \neq i} e(S, a'_j)) = \sum_{i=1}^{m} E(S - \{a'_i\}) - (m-1)E(S)$.

Let $\hat{a}$ be the additional agent in $\hat{S}$ ($\hat{S} - S = \{\hat{a}\}$). If $\hat{a}$ has a higher initial marginal value than a'_m, then the agents with the m highest initial marginal values in $\hat{S}$ are $a'_1, \ldots, a'_{m-1}$ and $\hat{a}$. It follows that $VCG(\hat{S}) = \sum_{i}^{m-1} E(\hat{S} - \{a'_i\}) + E(\hat{S} - \{\hat{a}\}) - (m-1)E(\hat{S}) = \sum_{i}^{m-1} E(\hat{S} - \{a'_i\}) + E(S) - (m-1)E(\hat{S})$. By Claim 13, for $i = 1, \ldots, m-1$, $E(\hat{S} - \{a'_i\}) - E(\hat{S}) \geq E(S - \{a'_i\}) - E(S)$. Hence, we have $VCG(\hat{S}) = \sum_{i}^{m-1} E(\hat{S} - \{a'_i\}) + E(S) - (m-1)E(\hat{S}) \geq \sum_{i}^{m-1} E(S - \{a'_i\}) - (m-1)E(S) + E(S) \geq \sum_{i}^{m-1} E(S - \{a'_i\}) - (m-1)E(S) + E(S - a'_m) = VCG(S)$.

If $\hat{a}$ has a lower initial marginal value than a'_m, the agents with the m highest initial marginal values in $\hat{S}$ would still be $a'_1, \ldots, a'_m$. By Claim 13, we have $VCG(\hat{S}) = \sum_{i}^{m} E(\hat{S} - \{a'_i\}) - (m-1)E(\hat{S}) = \sum_{i}^{m-1} E(\hat{S} - \{a'_i\}) - (m-1)E(\hat{S}) + E(\hat{S} - \{a'_m\}) \geq \sum_{i}^{m-1} E(S - \{a'_i\}) - (m-1)E(S) + E(\hat{S} - \{a'_m\}) \geq \sum_{i}^{m-1} E(S - \{a'_i\}) - (m-1)E(S) + E(S - \{a'_m\}) = VCG(S)$. ■

Claim 15 *For any $S \in \mathcal{P}(A)$, $0 \leq i \leq |S| - m - 2$, and $m + i + 2 \leq j \leq |S|$, we have $R(S, i) = R(U(S, j), i)$.*

Proof: We prove this claim by induction on i. For $i = 0$ and $j \geq m + 2$, we have $R(S, i) = VCG(S) = VCG(U(S, j)) = R(U(S, j), i)$, because, as we noted earlier, the total VCG payment depends only on the agents with the highest $m + 1$ initial marginal values in S, so removing the jth agent does not change the total VCG payment. Let us assume that we have proven that for $i = k$, if $j \geq m + k + 2$, $R(S, k) = R(U(S, j), k)$. Now let us consider the case where $i = k + 1$. By definition, $R(S, k + 1) = \frac{1}{m+k+1} \sum_{l=1}^{m+k+1} R(U(S, l), k)$. When $j \geq m + i + 2 = m + k + 3$, we can use the induction assumption (using the fact that $j - 1 \geq m + k + 2$) to show that $R(U(S, l), k) = R(U(U(S, l), j - 1), k)$. Hence, $R(S, k + 1) = \frac{1}{m+k+1} \sum_{l=1}^{m+k+1} R(U(S, l), k) = \frac{1}{m+k+1} \sum_{l=1}^{m+k+1} R(U(U(S, l), j - 1), k) = \frac{1}{m+k+1} \sum_{l=1}^{m+k+1} R(U(U(S, j), l), k) = R(U(S, j), k+1)$. (In the second-to-last step, the same agents are removed in a different order, although the agents' indices change as other agents are removed.) Hence the claim is also true for $i = k + 1$. ■

Claim 16 *For any $S, \hat{S} \in \mathcal{P}(A)$, $0 \leq i \leq |S| - m$, if $S \subseteq \hat{S}$, then $R(S, i) \leq R(\hat{S}, i)$. That is, R is nondecreasing in agents.*

Proof: We prove this claim by induction on i. When $i = 0$, using Claim 14, $R(S, i) = VCG(S) \leq VCG(\hat{S}) = R(\hat{S}, i)$. Let us assume that we have proven that the claim is true for $i = k$, that is, $R(S, k) \leq R(\hat{S}, k)$ if $S \subseteq \hat{S}$. Now let us consider the case where $i = k + 1$. If $\hat{S}$ and S are the same, the claim is trivial. Now suppose that $\hat{S}$ has one more agent than S, and that this additional agent has the qth highest initial marginal value in $\hat{S}$. If $q \geq m + k + 2$, $U(S, j) \subseteq U(\hat{S}, j)$ for all $j \leq m + k + 1$. By the induction assumption, we have $R(\hat{S}, k + 1) = \frac{1}{m+k+1} \sum_{j=1}^{m+k+1} R(U(\hat{S}, j), k) \geq \frac{1}{m+k+1} \sum_{j=1}^{m+k+1} R(U(S, j), k) = R(S, k + 1)$.

If $q \leq m + k + 1$, $U(S, j) \subseteq U(\hat{S}, j)$ for $j \leq q - 1$, and $U(S, j - 1) \subseteq U(\hat{S}, j)$ for $q + 1 \leq j \leq m + k + 1$. Using the induction assumption, we have $R(\hat{S}, k + 1) = \frac{1}{m+k+1} \sum_{j=1}^{m+k+1} R(U(\hat{S}, j), k) = \frac{1}{m+k+1} \sum_{j=1}^{q-1} R(U(\hat{S}, j), k) + \frac{1}{m+k+1} \sum_{j=q+1}^{m+k+1} R(U(\hat{S}, j), k) + \frac{1}{m+k+1} R(U(\hat{S}, q), k) \geq \frac{1}{m+k+1} \sum_{j=1}^{q-1} R(U(S, j), k) + \frac{1}{m+k+1} \sum_{j=q+1}^{m+k+1} R(U(S, j-1), k) + \frac{1}{m+k+1} R(S, k) \geq \frac{1}{m+k+1} \sum_{j=1}^{m+k} R(U(S, j), k) + \frac{1}{m+k+1} R(U(S, m + k + 1), k) = R(S, k + 1)$.

So, if $\hat{S}$ has one more element than S, then $R(S, k+1) \leq R(\hat{S}, k+1)$. It naturally follows that if $\hat{S}$ has even more elements, then we still have $R(S, k+1) \leq R(\hat{S}, k+1)$. ■

Claim 17 *For any $S \in \mathcal{P}(A)$, $R(S, i)$ is nonincreasing in i. In particular, setting $S = A$, V_i is nonincreasing in i.*

Proof: Using Claim 16, $R(S, i + 1) = \frac{1}{m+i+1} \sum_{j=1}^{m+i+1} R(U(S, j), i) \leq \frac{1}{m+i+1} \sum_{j=1}^{m+i+1} R(S, i) = R(S, i)$. ■

Claim 18 *For $0 \leq i \leq n - m - 1$, $\sum_{j=1}^{n} R(A_{-j}, i) = (n - m - 1 - i)V_i + (m + 1 + i)V_{i+1}$.*

Proof: Using Claim 15, we have $\sum_{j=1}^{n} R(A_{-j}, i) = \sum_{j=1}^{m+i+1} R(A_{-j}, i) + \sum_{j=m+i+2}^{n} R(A_{-j}, i) = (m+i+1)R(A, i+1) + (n-m-i-1)R(A, i) = (m+i+1)V_{i+1} + (n-m-i-1)V_i$. ■

Now that we have established these basic properties of R, we are ready to introduce the generalization of the worst-case optimal redistribution mechanism (both with or without deficits) to the setting where agents have nonincreasing marginal values over units.

Theorem 5 *When agents have nonincreasing marginal values over units, for any m and n with $n \geq m+2$, the worst-case optimal redistribution fraction (without deficits) is*

$$k^* = 1 - \frac{\binom{n-1}{m}}{\sum_{j=m}^{n-1} \binom{n-1}{j}}$$

(the same as in Theorem 1), and the worst-case imbalance fraction (with deficits) is

$$k_d^* = \frac{\binom{n-1}{m}}{\sum_{j=m+1}^{n} \binom{n}{j}}$$

(the same as in Theorem 3).

In each case, the following is a worst-case optimal mechanism: to agent a_i, redistribute $\frac{1}{m}\sum_{j=m+1}^{n-1} c_j^* R(A_{-i}, j-m-1)$. Here, the c_j^* from Theorem 1 are used to maximize the worst-case redistribution fraction without deficits, and the c_j^* from Theorem 3 are used to minimize the worst-case imbalance fraction when deficits are allowed. The mechanisms obtained in this way in fact generalize the mechanisms from Theorem 1 and Theorem 3.

Proof: In each case, the mechanism is strategy-proof because each agent's redistribution payment is independent of her own bid (A_{-i} does not contain a_i). It is deterministic, efficient and anonymous. Because $R(A_{-i}, j-m-1)$ is nonincreasing in j, and $\sum_{j=m+1}^{i} c_j^* \geq 0$ for $i = m+1, \ldots, n-1$, it follows by Lemma 1 that the mechanism is also individually rational.

Now, we recall that in the unit demand setting, for any bid vector $\hat{v}_1 \geq \hat{v}_2 \geq \ldots \geq \hat{v}_n$, the total amount redistributed by the worst-case optimal mechanism is $\sum_{j=m+1}^{n-1} c_j^*((n-j)\hat{v}_j + j\hat{v}_{j+1})$, which is always at least $k^* m\hat{v}_{m+1}$ and at most $m\hat{v}_{m+1}$ when we use the c_j^* from Theorem 1; and which is always at least $(1-k_d^*)m\hat{v}_{m+1}$ and at most $(1+k_d^*)m\hat{v}_{m+1}$ when we use the c_j^* from Theorem 3. We next show that analogous bounds apply to the more general mechanisms, which will complete the proof.

For the more general mechanisms, the total redistribution payment is $\frac{1}{m}\sum_{i=1}^{n}\sum_{j=m+1}^{n-1} c_j^* R(A_{-i}, j-m-1) = \frac{1}{m}\sum_{j=m+1}^{n-1} c_j^* \sum_{i=1}^{n} R(A_{-i}, j-m-1) = \frac{1}{m}\sum_{j=m+1}^{n-1} c_j^*((n-j)V_{j-m-1} + jV_{j-m})$. This expression is very similar to the total redistributed by the mechanisms in the unit demand setting: the only differences are that each $\hat{v}_j$ has been replaced by the V_{j-m-1}, and there is an additional factor $\frac{1}{m}$. Now, the bounds for the unit demand setting hold for *any* nonincreasing sequence of $\hat{v}_j$;

and, by Claim 17, we have $V_0 \geq V_1 \geq \ldots \geq V_{n-m-1}$. Hence, $\frac{1}{m}\sum_{j=m+1}^{n-1} c_j^*((n-j)V_{j-m-1} + jV_{j-m})$ is in $[k^*V_0, V_0]$ when we use the c_j^* from Theorem 1, and in $[(1-k_d^*)V_0, (1+k_d^*)V_0]$ when we use the c_j^* from Theorem 3. Because $V_0 = R(A, 0) = VCG(A)$ is the total VCG payment, this proves the result. ∎

So far we have only talked about the case where $n \geq m+2$. For the purpose of completeness, we provide the following claim for the $n \leq m+1$ case.

Claim 19 *For any m and n with $n \leq m+1$, the original VCG mechanism (that is, redistributing nothing) is worst-case optimal, both with or without deficits, among all redistribution mechanisms that are deterministic, anonymous, strategy-proof and efficient.*

Proof: Suppose there is a mechanism that satisfies all the desirable properties and has a worst-case performance that is at least as good as the VCG mechanism. Because the mechanism is strategy-proof, the redistribution payment received by an agent should be independent of her own bid. Also, if a bid profile results in a total VCG payment of 0, then under this profile, the total redistribution payment must also be 0. (If the objective is to maximize redistribution without deficits, negative total redistribution would result in worse performance than VCG, and positive redistribution would violate the non-deficit constraint. If the objective is to minimize imbalance, either negative or positive redistribution would result in worse performance than VCG. These arguments are analogous to those in the proofs of Claim 7 and Claim 12.)

For the purpose of this proof only, we introduce the following notations. If an agent has marginal value 1 for every unit among the first k units, and 0 for any further units, we denote her bid by k. These are the only bids that we will use in this proof. For $b_i \in \mathbf{N}$, let $f(b_1, b_2, \ldots, b_{n-1})$ be the redistribution payment received by an agent if the other agents' bids are $b_1, \ldots, b_{n-1}$.

We will prove that for any set of nonnegative integers $b_1, b_2, \ldots, b_{n-1}$, if $\sum_{i=1}^{n-1} b_i \leq m$, we have $f(b_1, \ldots, b_{n-1}) = 0$. We will do so by proving by induction on k ($k \leq m$) the claim that for any set of nonnegative integers $b_1, b_2, \ldots, b_{n-1}$, if $\sum_{i=1}^{n-1} b_i \leq k$, we have $f(b_1, \ldots, b_{n-1}) = 0$.

For the case $k = 0$, let us consider the case where all the agents bid 0, so that the total redistribution payment is $nf(0, 0, \ldots, 0)$. Because the total VCG payment is 0, the total redistribution must be 0, therefore $f(0, 0, \ldots, 0)$ must be 0.

Now let us assume that for any set of nonnegative integers $b_1, b_2, \ldots, b_{n-1}$, if $\sum_{i=1}^{n-1} b_i \leq k$, we have $f(b_1, \ldots, b_{n-1}) = 0$. Let $b'_1, b'_2, \ldots, b'_{n-1}$ be any set of nonnegative integers that satisfies $\sum_{i=1}^{n-1} b'_i = k+1$. Consider the bid profile (consisting of n bids) formed by the b'_i and one 0. The redistribution payment received by the agent that bids 0 is then $f(b'_1, b'_2, \ldots, b'_{n-1})$. We note that some of the b'_i may equal 0 as well; by anonymity, the payment for these agents should be the same. The redistribution payment received by any agent that does not bid 0 is 0 by the induction assumption. Hence, the total redistribution is a positive multiple of $f(b'_1, b'_2, \ldots, b'_{n-1})$. Given that $k+1 \leq m$, the total VCG payment is 0, so it must be that $f(b'_1, b'_2, \ldots, b'_{n-1}) = 0$, completing the proof by induction.

Having proved this, we now find an example with positive total VCG payment but zero total redistribution, which will complete the proof. We recall $m \geq n-1$. Let us consider the bid profile where one agent bids $m-n+2$ and the other agents each bid 1. Then, the total redistribution payment is $(n-1)f(m-n+2,\underbrace{1,\ldots,1}_{n-2}) + f(\underbrace{1,\ldots,1}_{n-1}) = 0$ (since the previous claim applies to both $f(m-n+2,\underbrace{1,\ldots,1}_{n-2})$ and $f(\underbrace{1,\ldots,1}_{n-1})$). However, the total VCG payment is positive. Hence, the mechanism has a redistribution fraction of 0% and an imbalance fraction of 100% on this instance. ■

11 General Multi-Unit Auctions

In Section 10, we showed how the results for the unit demand setting can be generalized to the setting where agents have nonincreasing marginal values over the units. The natural next question is whether they can be generalized even further. In this section, we study multi-unit settings without any constraint on the bidders' valuations—that is, marginal values can be increasing (but they cannot be negative: units can always be freely disposed of). We show that when there are at least two units, the original VCG mechanism (that is, redistributing nothing) is worst-case optimal, both with and without deficits. (When there is only a single unit, then the agents must have unit demand, so the previous results do apply.)

Claim 20 *In multi-unit auctions without any restrictions on agents' valuations, when the number of units m is at least 2, the original VCG mechanism (that is, redistributing nothing) is worst-case optimal, both with or without deficits, among all redistribution mechanisms that are deterministic, anonymous, individually rational, strategy-proof and efficient.*

We emphasize that unlike some of the earlier proofs in this paper, this proof does require individual rationality.

Proof: Claim 19 already established that for $n-2<m$, the original VCG mechanism is worst-case optimal even when we do assume nonincreasing marginal values, so it suffices to consider only the case where $n-2 \geq m$. Suppose there is a mechanism that satisfies all the desirable properties and has a worst-case performance that is at least as good as the original VCG mechanism. Because the mechanism is strategy-proof, the redistribution payment received by an agent should be independent of her own bid.

Also, if a bid profile results in a total VCG payment of 0, then under this profile, the total redistribution payment must also be 0 (otherwise, the performance is worse than that of the original VCG mechanism).

For the purpose of this proof only, we introduce the following notations. If an agent has marginal value 0 for every unit among the first $m-1$ units, and marginal value 1 for the mth unit, we denote her bid by B_1. If an agent has marginal value 1 for the first unit, and 0 for any further units, we denote her bid by B_2. If an agent has marginal value 0 for all units, we denote her bid by 0. These are the only bids that we will use in this proof. For $b_i \in \{B_1, B_2, 0\}$, let $f(b_1, b_2, \ldots, b_{n-1})$ be the redistribution payment received by an agent if the other agents' bids are $b_1, \ldots, b_{n-1}$. We need $f(b_1, b_2, \ldots, b_{n-1}) \geq 0$ to ensure individual rationality.

We will prove the following:

- $f(0, 0, \ldots, 0) = 0$
- $f(B_1, 0, \ldots, 0) = 0$
- $f(B_2, 0, \ldots, 0) = 0$
- $f(B_1, B_2, 0, \ldots, 0) = 0$

For $f(0, 0, \ldots, 0)$, let us consider the case where all the agents bid 0, so that the total redistribution payment is $nf(0, 0, \ldots, 0)$. Because the total VCG payment is 0, the total redistribution must be 0, therefore $f(0, 0, \ldots, 0)$ must be 0.

For $f(B_1, 0, \ldots, 0)$, let us consider the case where one agent bids B_1 and all the other agents bid 0, so that the total redistribution payment is $(n-1)f(B_1, 0, \ldots, 0) + f(0, 0, \ldots, 0) = (n-1)f(B_1, 0, \ldots, 0)$. Because the total VCG payment is 0, the total redistribution must be 0, therefore $f(B_1, 0, \ldots, 0)$ must be 0. The same argument can be used to show that $f(B_2, 0, \ldots, 0) = 0$.

For $f(B_1, B_2, 0, \ldots, 0)$, let us consider the case where one agent bids B_1, two agents bid B_2 and all the other agents bid 0, so that the total redistribution payment is $(n-3)f(B_1, B_2, B_2, 0, \ldots, 0) + 2f(B_1, B_2, 0, \ldots, 0) + f(B_2, B_2, 0, \ldots, 0)$. However, the total VCG payment is still 0 for these bids (the agents that bid B_2 win; if one of them is removed, we can do no better than to still allocate one unit to the other B_2 agent, and nothing to the other agents—hence each B_2 agent pays 0). Hence, the total redistribution must be 0. Because f is nonnegative everywhere, it follows that $f(B_1, B_2, 0, \ldots, 0)$ must equal 0.

Having proved this, we now find an example with positive total VCG payment but zero total redistribution, which will complete the proof. Let us consider the bid profile where one agent bids B_1, one agent bids B_2, and the other agents all bid 0. Then, the total redistribution payment is $(n-2)f(B_1, B_2, 0, \ldots, 0) + f(B_1, 0, \ldots, 0) + f(B_2, 0, \ldots, 0) = 0$. However, the total VCG payment is positive (because we can accept at most one of the B_1 bid and the B_2 bid). Hence, the mechanism has a redistribution fraction of 0% and an imbalance fraction of 100% on this instance. ∎

12 Conclusions

For allocation problems with one or more items, the well-known Vickrey-Clarke-Groves (VCG) mechanism (also known as the Clarke mechanism or the Generalized Vickrey Auction) is efficient, strategy-proof, individually rational, and does not incur a deficit. However, the VCG mechanism is not (strongly) budget balanced: generally, the agents' payments will sum to more than 0. If there is an auctioneer who is selling the items, this may be desirable, because the surplus payment corresponds to revenue for the auctioneer. However, if the items do not have an owner and the agents are merely interested in allocating the items efficiently among themselves, any surplus payment is undesirable, because it will have to flow out of the system of agents. In 2006, Cavallo [4] proposed a mechanism that redistributes some of the VCG payment back to the agents, while maintaining efficiency, strategy-proofness, individual rationality, and the non-deficit property. In this paper, we extended Cavallo's technique in a restricted setting. We studied allocation settings where there are multiple indistinguishable units of a single good, and the agents have nonincreasing marginal values. (For this specific setting, Cavallo's mechanism coincides with a mechanism proposed by Bailey in 1997 [3].) We first considered the simpler unit demand setting. We proposed a family of mechanisms that redistribute some of the VCG payment back to the agents. All mechanisms in the family are efficient, strategy-proof, individually rational, and never incur a deficit. The family includes the Bailey-Cavallo mechanism as a special case. We then provided an optimization model for finding the optimal mechanism—that is, the mechanism that maximizes redistribution in the worst case—inside the family, and showed how to cast this model as a linear program. We gave both numerical and analytical solutions of this linear program, and the (unique) resulting mechanism shows significant improvement over the Bailey-Cavallo mechanism (in the worst case). We proved that the obtained mechanism is worst-case optimal among *all* anonymous deterministic mechanisms that satisfy the above properties. Using similar techniques, we also found the worst-case optimal mechanism when deficits are allowed. We generalized both mechanisms to the setting where the agents do not necessarily have unit demand, but do have nonincreasing marginal values over units. In each case, the worst-case performance of the generalized mechanism is the same as in the unit demand setting, and hence the generalized mechanisms are also worst-case optimal. Finally, for multi-unit auctions without any restriction on agents' valuations, we showed a negative result: no mechanism is better than the original VCG mechanism in the worst case.

Incidentally, all of our results can also be applied to multi-unit *reverse* auctions, in which a single buyer needs to procure m units from n potential sellers (agents). (We can also view units as tasks that need to be performed by the agents.) For example, consider the setting in which each agent has an obligation to supply one unit (perform one task), but $m < n$, that is, not every unit is actually needed. In this case, we can run a forward auction for the $n-m$ rights not to supply a unit. Hence, all of our results hold with m replaced by $n - m$. This example is analogous to the unit demand setting, but our results can also be applied to more general valuation functions. We note, however, that this prior-obligation view corresponds to a different notion of individual rationality than the one typically used in reverse auctions.

One direction for future research is to extend these results to combinatorial auctions (with distinguishable items). Another direction is to consider objectives that are not worst-case. Yet another direction is to consider whether this mechanism has applications to collusion. For example, in a typical collusive scheme, there is a *bidding ring* consisting of a number of colluders, who submit only a single bid [13, 22]. If this bid wins, the colluders must allocate the item amongst themselves, perhaps using payments—but of course they do not want payments to flow out of the ring.

This work is part of a growing literature on designing mechanisms that obtain good results in the worst case. Traditionally, economists have mostly focused either on designing mechanisms that always obtain certain properties (such as the VCG mechanism), or on designing mechanisms that are optimal with respect to some prior distribution over the agents' preferences (such as the Myerson auction [25] and the Maskin-Riley auction [23] for maximizing expected revenue). Some more recent papers have focused on designing mechanisms for profit maximization using worst-case competitive analysis (*e.g.* [12, 1, 19, 11]). There has also been growing interest in the design of *online* mechanisms [10] where the agents arrive over time and decisions must be taken before all the agents have arrived. Such work often also takes a worst-case competitive analysis approach [18, 17]. It does not appear that there are direct connections between our work and these other works that focus on designing mechanisms that perform well in the worst case. Nevertheless, it seems likely that future research will continue to investigate mechanism design for the worst case, and hopefully a coherent framework will emerge.

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